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RANKS AND SUBDEGREES OF $G = PGL(2,q)$ ON THE COSETS OF ITS SUBGROUPS

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Abstract- This paper is set to compute the ranks and the subdegrees of $PGL(2,q)$ acting on the cosets of its subgroups namely; C_{q-1}, C_{q+1} and P_q. This is done using the method proposed by Ivanov *et al*. which uses marks of a permutation group. For the action of $PGL(2,q)$ on the cosets of C_{q-1} the subdegrees are shown to be 1^2 and $(q-1)^{(q+2)}$ and the rank is $q+4$. For P_q the subdegrees are $1^{(q-1)}$ and $q^{(q-1)}$ and the rank is $2(q-1)$.

Index Terms- Rank,Subdegrees

I. INTRODUCTION

a) Burnside's definition of marks

Let P_G be a permutation representation (transitive or intransitive) of a group G on *X*. The mark of the subgroup *H* of *G* in P_G is the number of points of *X* fixed by every permutation of *H*. incase $G(/G_i)$ is a coset representation ; $m(G_j,G_i,G)$, the mark of G_j in $G(/G_i)$ is the number of cosets of G_i in G left fixed by every permutation of G_j.

b) White's definition of marks;

$$m(G_j,G_i,G) = \frac{1}{|G_i|}\sum_{g\in G} x(g^{-1}G_j g \subseteq G_i),$$

where$x$ (statement)

$$= \begin{cases} 1, & \textit{if the statement is true} \\ 0, & \textit{if the statement is untrue} \end{cases}$$

a) Ivanov *et al*. definition of marks

Ivanov *et al*. (1983) defined the mark in terms of normalizers of subgroups of a group as; if $G_j \le G_i \le G$ and $(G_{j1}, G_{j2}, \dots\dots\dots, G_{jn})$ is a complete set of conjugacy class representatives of subgroups of G_i that are conjugate to G_j in G, then;

$$m(G_j,G_i,G) = \sum_{k=1}^{n} |N_G(G_{j_k}) : N_{Gi}(G_{j_k})|.$$

In particular when $n=1$, G_j is conjugate in G_i to all subgroups $G_{j'}$ that are contained in G_i and are conjugate to G_j in *G* and;

$$m(G_j,G_i,G) = |N_G(G_j) : N_{G_i}(G_j)|.$$

It can be shown that these definitions are equivalent. (Kamuti,1992)

A. Lemma

Let C_d (d coprime to p) be a cyclic subgroup of order d, then;

$$N_G(C_d) = \begin{cases} D_{q\pm1,} & p\ odd \\ D_{2\,(q\pm1)} & p=2 \end{cases}$$

$\pm$ sign as $d|q\pm1$.
(Dickson, 1901)

B. Lemma

Let C_p be a cyclic subgroup of order *p* in *G*, then

$$N_G(C_p) = \begin{cases} \frac{1}{2}q(p-1) & p\ odd, f\ odd \\ q(p-1) & p\ odd\ f\ even \\ q & p=2 \end{cases}$$

(Dickson, 1901)

C. Lemma

Let $d>3$ be a divisor of $(q\pm1)$ and δ be the quotient, then

$$N_G(D_{2d}) = \begin{cases} D_{2d,} & if\ \delta\ is\ odd \\ D_{4d} & if\ \delta\ is\ even \end{cases}$$

(Dickson, 1901)

D. Theorem

The number Q_i satisfy the system of equations;

$$\sum_{i=j}^{t} Q_i m(H_j,H_i,H) = m(H_j,H,G)$$

for each $j = 1,2,\dots\dots,t$.
(Kamuti, 1992)

II. SUBDEGREES OF G ON THE COSETS OF $H = C_{q-1}$

Since *H* is abelian, each of its subgroups is normal. Suppose *H* has s subgroups say,
$H_1 = 1$, $H_2, H_3, \dots\dots\dots\dots, H_s = H$ with $i|q-1$ and, $(i = 1,2,\dots, s-1)$.

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Now using the method proposed by Ivanov et al (1983) the table of marks of *H* can be computed as follows;

Table 2.1: Table of marks of $\boldsymbol{H = C_{q-1}}$

	H_1	$H_2 \dots \dots \dots \dots$	H_s
$H(/H_1)$	$q-1$		
$H(/H_2)$	m_{21}	m_{22}	
….	………	.	
….	… . …….		
$H(/H_s)$	1	1 $\dots \dots \dots$	1

After computing the table of marks of *H*, we now proceed to find
$m(F) = m(F,H,G)$, where *F* is a representative of a conjugacy class in *H* and $F \leq H$. The value of $m(F)$ is obtained using Lemma1.1 and the method proposed by Ivanov et al. (1983). The values of $m(F)$ are displayed in Table 2.2 below.

Table 2.2: The mark of *F* where $\boldsymbol{F \leq H = C_{q-1}}$

F	$H_1 = I$	H_2	H_3	H_{4}	…	H_{n-1}	H_n
$M(F)$	$q(q+1)$	2	2	2	…	2	2

Let $Q = (Q_1, Q_2, Q_3, \dots \dots, Q_{S-1}, Q_S)$ denote the number of suborbits Δ_j. Then by Theorem 1.4 and using Table 2.2 and Table 2.1 we obtain the following system of equations;

$$(q-1)Q_1 + m_{21}Q_2 + \cdots + m_{s-11}Q_{s-1} + Q_s = q(q+1)$$
$$m_{22}Q_2 + \cdots + m_{s-12}Q_{s-1} + Q_s = 2$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$m_{s-1s-1}Q_{s-1} + Q_s = 2$$
$$Q_s = 2.$$

Solving the above system of equations we obtain
$Q = (q+2, 0, 0, \dots \dots, 2)$.
Hence the subdegrees of *G* on the cosets of *H* are shown in Table 2.3 below

Table 2.3: Subdegrees of *G* on the cosets of $\boldsymbol{C_{q-1}}$

Suborbit Length	1	$q-1$
No. of Suborbits	2	$q+2$

Therefore the rank (r) is given by;
$r = 2 + q + 2 = q + 4$

III. SUBDEGREES OF $\boldsymbol{G}$ ON THE COSETS OF $\boldsymbol{H = C_{q+1}}$

Since *H* is cyclic thus it is abelian, so each of its subgroups is normal. If H has s subgroups say, $H_1 = 1$, $H_2, H_3, \dots \dots \dots, H_s = H$ with $i | q+1$ and $(i = 1, 2, \dots, s-1)$.

Again using the method proposed by Ivanov et al. (1983) the table of marks of *H* can be as shown below;

Table 3.1: Table of marks of $\boldsymbol{H = C_{q+1}}$

	H_1	$H_2 \dots \dots \dots \dots$	H_s
$H(/H_1)$	$q+1$		
$H(/H_2)$	m_{21}	m_{22}	
….	…	…. …	
….	…	…. ………	
$H(/H_s)$	1	1 $\dots \dots \dots$	1

Using Lemma 1.1 and Lemma 1.2 we find
$m(F) = (q(q-1), 2, 2, \dots, 2)$, s −tuples. Let
$Q = (Q_1, Q_2, Q_3, \dots \dots \dots, Q_s)$. Then using Table 3.1 and Theorem 1.4 we form the following systems of equations;

$$(q+1)Q_1 + m_{21}Q_2 + \cdots + m_{s-11}Q_{s-1} + Q_s = q(q-1)$$
$$\dots \dots \dots \dots \dots \dots \dots \dots$$
$$m_{s-1s-1}Q_{s-1} + Q_s = 2$$
$$Q_s = 2$$

Solving the above system of equations we obtain
$Q = (q-2, 0, 0, \dots, 0, 2)$.
Hence the subdegrees of *G* on the cosets of *H* are shown in Table 3.2 below

Table 3.2: Subdegrees of *G* on the cosets of $\boldsymbol{H = C_{q+1}}$

Suborbit Length	1	$q+1$
No. of Suborbits	2	$q-2$

Therefore the rank (r) is given by;
$r = 2 + q - 2 = q$

IV. SUBDEGREES OF $\boldsymbol{G}$ ON THE COSETS OF $\boldsymbol{H = P_q}$

Suppose *H* has *n* subgroups say
$H_1 = I,\ H_2, H_3, H_4, \dots \dots \dots \dots H_n = H$.
These subgroups are of order $1, p, p^2, p^3, \dots \dots \dots, p^n$ respectively since *H* is a *p*-group. Also all these subgroups are normal in *H*. So the table of marks of *H* omitting the zeros above the main diagonal is as shown in Table 4.1 below.

Table 4.1: Table of marks of $\boldsymbol{H = P_q}$

	$H_1 = I$	H_2	H_3	$H_4 \dots \dots \dots \dots \dots \dots$	H_{n-1}	$H_n = H.$
$H(/H_1)$	$q = p^n$					
$H(/H_2)$	p^{n-1}	p^{n-1}				
$H(/H_3)$	p^{n-2}	p^{n-2}	p^{n-2}			
$H(/H_4)$	p^{n-3}	p^{n-3}	p^{n-3}	p^{n-3}		
	………………………………					
……..	…………………………………………					
	p	p	p	$p \dots \dots$		p
……..						

$H(/H_{n-1})$				
$H(/H_n)$	1	1	………..	1

After computing the table of marks of *H*, we now proceed to find
$m(F) = m(F, H, G)$, where F is a representative of a conjugacy class in H and $F \leq H$. The values of $m(F)$ are displayed in Table 4.2 below.

Table 4.2:The mark of F, where $\boldsymbol{F \leq H = P_q}$

F	$H_1 = I$	H_2	H_3	H_4	……	H_{n-1}	$H_n = H$
$M(F)$	$q^2 - 1$	$q - 1$	$q - 1$	$q - 1$	…	$q - 1$	$q - 1$

Let $Q = (Q_1, Q_2, Q_3, \dots\dots, Q_{n-1}, Q_n)$ denote the number of suborbits Δ_j. Then by Theorem 1.4 and using Table 4.1 and Table 4.2 we obtain the following system of equations;

$$p^n Q_1 + p^{n-1} Q_2 + P^{n-2} Q_3 + P^{n-3} Q_4 + \cdots \dots\dots + pQ_{n-1} + Q_n = q^2 - 1$$
$$p^{n-1} Q_2 + P^{n-2} Q_3 + P^{n-3} Q_4 + \cdots \dots\dots + pQ_{n-1} + Q_n = q - 1$$
$$P^{n-2} Q_3 + P^{n-3} Q_4 + \cdots \dots\dots + pQ_{n-1} + Q_n = q - 1$$

……………………………………………

$$pQ_{n-1} + Q_n = q - 1$$
$$Q_n = q - 1$$

Solving the above systems of equation we obtain
$Q = (q - 1, 0, 0, \dots\dots, 0, q - 1)$.
Hence the subdegrees of *G* on the cosets of *H* are shown in +Table 4.3 below.

Table 4.3: Subdegrees of $\boldsymbol{G}$ on the cosets of $\boldsymbol{H = P_q}$

Suborbit Length	1	q
No. of Suborbits	$q - 1$	$q - 1$

Therefore the rank (r) is given by;
$r = (q - 1) + (q - 1) = 2(q - 1)$

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