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Series of New Divergence Measures and Relation to Other Well Known Divergence Measures

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***Abstract*— There are several types of divergence measures used in measuring the distance or affinity between two probability distributions. In this paper we derive some families of divergence measures using properties of convex function f, defined on (0, ∞) and Csiszar's f-divergence measure. Some relations among new and other well-known divergence measures are also obtained.**

Index Terms—Triangular discriminations Csiszar's f-divergence,chi-square divergence measure etc.

***Mathematics Subject Classification*- 94A17, 26D15.**

I. Introduction

Let $\Gamma_n = \{P = (p_1, p_2, p_3, \ldots p_n)$: $p_i > 0, \sum_{i=1}^{n} p_i = 1\}, n \geq 2$

Be the set of all complete finite discrete probability distributions. Csiszar's [1], given the generalized f-divergence measure, which is given by

$C_f(P,Q) = \sum_{i=1}^{n} q_i f(\frac{p_i}{q_i})$...(1.1)

Where f : $(0,\infty) \to$ R (set of real numbers) is real, continuous and convex function and P, Q ϵ Γn. Many well-known information divergence measures can be obtained from this divergence measure by defining the convex function f. some of those are as follows

- Chi-square divergence (Pearson [2])

$$\chi^2(P,Q) = \sum_{i=0}^{n} \frac{(p_i - q_i)^2}{(q_i)} \quad \ldots\ldots\ldots (1.2)$$

- Puri and Vineze divergence (Kafka, Osterreicher and Vineze) [5]

$$\Delta m(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{2m}}{(p_i + q_i)^{2m-1}}, \text{ m = 1, 2, 3} \quad \ldots\ldots (1.3)$$

Where

$$\Delta(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{(p_i + q_i)} \quad \ldots\ldots\ldots\ldots (1.4)$$

Triangular discrimination, is a particular case of (1.3) at m = 1

* Harmonic mean divergence [3]

$$H(P,Q) = \sum_{i=1}^{n} \frac{2 p_i q_i}{p_i + q_i} \ldots\ldots (1.5)$$

* Jain K.C. and Saraswat R.N. [4]

$$R_k^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{k+1}}{p_i^k} \exp\left\{\frac{(p_i - q_i)^2}{p_i^2}\right\}; \text{ k= 1, 3, 5,} \ldots (1.6)$$

- Hellinger Discrimination [6]

$$h(P,Q) = 1 - B(P,Q) = \frac{1}{2} \sum_{i=1}^{n} \left(\sqrt{p_i} - \sqrt{q_i}\right)^2 \quad \ldots\ldots\ldots\ldots (1.7)$$

$$\text{Where} B(P,Q) = \sum_{i=1}^{n} \sqrt{p_i q_i} \ldots\ldots\ldots\ldots\ldots (1.8)$$

is known as Bhattacharya divergence measure[7]

- Variational Distance [2]

$$V(P,Q) = \sum_{i=1}^{n} |p_i - q_i| \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.9)$$

In whole paper, we shall consider new function which is a convex and normalized. We shall derive some new information divergence measures in the terms of exponential function. In next section, we shall derive relations among new information divergence measures and well establish information divergence measures.

II. NEW INFORMATION DIVERGENCE MEASURE

In this section we shall find out the new information divergence measure with the help of the following convex function.

Let us consider the function

$f: (o, \infty) \to R$ Suchthat

$$f_k(t) = \frac{(t-1)^{2k+2}}{t^{2k}}; \text{ k = 1, 2, 3}\ldots\ldots\ldots\ldots (2.1)$$

Then

$$f_k'(t) = \frac{2(t-1)^{2k+1}(t+k)}{t^{2k+1}}; \ldots\ldots\ldots\ldots\ldots (2.2)$$

$$\text{And} f_k''(t) = \frac{2(t-1)^{2k}[(t+k)^2 + k^2 + k]}{t^{2k+2}} \ldots\ldots\ldots\ldots (2.3)$$

The function $f_k(t)$ is convex , since $f_k''(t) \geq 0, \forall t > 0 \; ; k = 1, 2, 3, \ldots\ldots$

And normalized also since f(1) = 0

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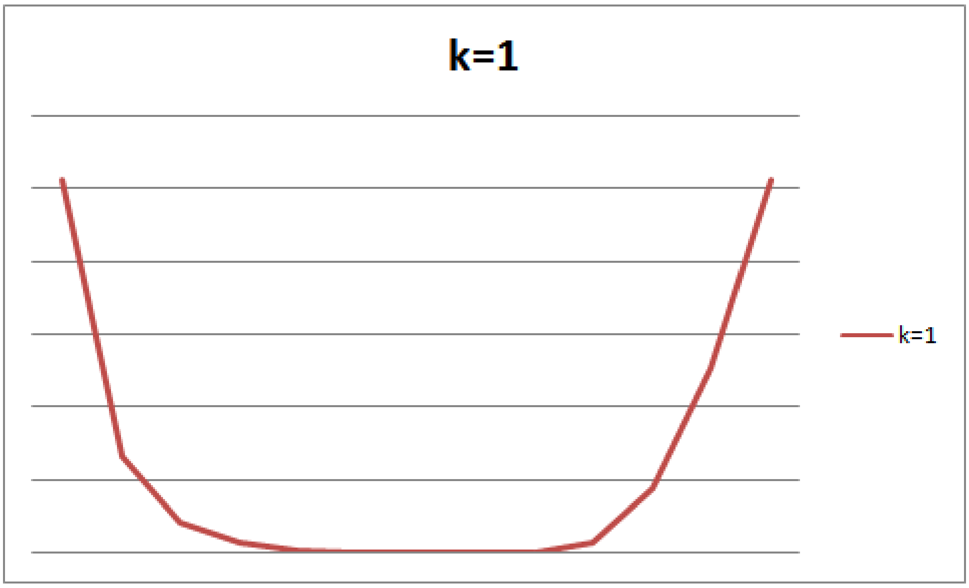

FIGURE:1

Figure 1. Shows the behavior of the function $f_k\ (t)$, is always canvex if k = 1, 2, 3 ...... $\forall t > o$.
Applying Csiszar's f divergence properties on equation (2.1), we get

$$J_k^c(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{2k+2}}{p_i^{2k} q_i}\ ; k = 1, 2, 3, \dots\dots\dots (2.4)$$

Divergence measure J_k^c(P, Q) is non-symetric divergence measure since

J_k^c(P, Q) ≠ J_k^c (Q, P) ...... (2.5)

Now, from equation (2.1) for k = 1, 2, 3, 4, 5 ...., we get the following convex functions

$$f_1(t) = \frac{(t-1)^4}{t^2}, f_2(t) = \frac{(t-1)^6}{t^4}, f_3(t) = \frac{(t-1)^8}{t^6}, f_4(t) = \frac{(t-1)^{10}}{t^8} \dots\dots\dots$$

We know that, sum of convex functions is also a convex function
i.e.

$$C_1 f_1(t) + C_2 f_2(t) + C_3 f_3(t) + +C_4 f_4(t) + \dots\dots\dots$$

is also a convex function, where $C_1, C_2, C_3, C_4, \dots\dots$ are arbitrary positive constants andat least one C_i(*i=1,2,3…*) is not equal to zero.
Now,

$$F_i(t) = \sum_{i=1}^{\infty} C_i f_i(t) = C_1 f_1(t) + C_2 F_2(t) + C_3 F_3(t) + C_4 F_4(t) + \dots\dots\dots$$

$$\Rightarrow F_i(t) = C_1 \frac{(t-1)^4}{t^2} + C_2 \frac{(t-1)^6}{t^4} + C_3 \frac{(t-1)^8}{t^6} + C_4 \frac{(t-1)^{10}}{t^8} + \dots \dots\dots (2.6)$$

Taking $C_1 = 1,\ C_2 = \frac{1}{1!}, C_3 = \frac{1}{2!}, C_4 = \frac{1}{3!}, \dots\dots$
Then

$$F_1(t) = 1.\frac{(t-1)^4}{t^2} + 1.\frac{(t-1)^6}{t^4} + \frac{1}{2!}\frac{(t-1)^8}{t^6} + \frac{1}{3!}\frac{(t-1)^{10}}{t^8} + \cdots$$

$$= \frac{(t-1)^4}{t^2}\left[1 + \left(\frac{(t-1)^2}{t^2}\right)^1 + \frac{1}{2!}\left(\frac{(t-1)^2}{t^2}\right)^2 + \frac{1}{3!}\left(\frac{(t-1)^2}{t^2}\right)^3 + \cdots\right]$$

$$\Rightarrow F_1(t) = \frac{(t-1)^4}{t^2} \exp\left\{\frac{(t-1)^2}{t^2}\right\} \dots\dots (2.7)$$

Now, for (2.7), Divergence measure of Csiszar'sf-divergence class

$$J_1^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^4}{p_i^2\, q_i} \exp\left\{\frac{(p_i - q_i)^2}{p_i^2}\right\} \ldots\ldots (2.8)$$

Next, taking

$$C_1 = 0, C_2 = 1, C_3 = \frac{1}{1!}, C_4 = \frac{1}{2!}, C_5 = \frac{1}{3!}, \ldots\ldots in (2.6)$$

We get

$$F_2(t) = \frac{(t-1)^6}{t^4} \exp\left\{\frac{(t-1)^2}{t^2}\right\} \ldots\ldots\ldots\ldots\ldots (2.9)$$

Hence, Divergence measure of Csiszar's f divergence class

$$J_2^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^6}{p_i^4\, q_i} \exp\left\{\frac{(p_i - q_i)^2}{p_i^2}\right\} \ldots\ldots (2.10)$$

Similarly, by appropriate selection of constants, we get the following convex functions

$$F_k(t) = \frac{(t-1)^{2k+2}}{t^{2k}} \exp\left\{\frac{(t-1)^2}{t^2}\right\} ; k = 1, 2, 3, \ldots\ldots\ldots (2.11)$$

And the corresponding series of divergence measures of Csiszar's of divergence class

$$J_k^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{2k+2}}{p_i^{2k} q_i} \exp\left\{\frac{(p_i - q_i)^2}{p_i^2}\right\} \ldots\ldots (2.12)$$

; k= 1, 2, 3, 4, …………..

Further $F_k(1)$ = o, so that $J_k^*(P,P) = 0$ and the convexity of the function $F_k(t)$ ensure that the measure $J_k^*(P,Q)$ is non-negative.

Thus we can say that the measure (2.12) is non-negative and convex in the pair of probability distributions (P, Q) $\epsilon\ \Gamma_n$. Since

$$J_1^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^4}{p_i^2\, q_i} \exp\left\{\frac{(p_i - q_i)^2}{p_i^2}\right\} \ldots (2.13)$$

$$J_2^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^6}{p_i^4\, q_i} \exp\left\{\frac{(p_i - q_i)^2}{p_i^2}\right\} \ldots\ldots (2.14)$$

$$J_3^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^8}{p_i^6\, q_i} \exp\left\{\frac{(p_i - q_i)^2}{p_i^2}\right\} \ldots\ldots (2.15)$$

And so on.

III. SOME OTHER NEW INFORMATION DIVERGENCE MEASURES

In this section, we shall derive some new divergence measures using the property of convex functions, since the sum of convex functions is again a convex function, therefore we have the following convex functions

$$\frac{(t-1)^4}{t^2} + \frac{(t-1)^6}{t^4} = \frac{(t-1)^4(2t^2-2t+1)}{t^4} \quad \ldots\ldots\ldots\ldots (3.1)$$

$$\frac{(t-1)^6}{t^4} + \frac{(t-1)^8}{t^6} = \frac{(t-1)^6(2t^2-2t+1)}{t^6} \quad \ldots\ldots\ldots\ldots (3.2)$$

$$\frac{(t-1)^8}{t^6} + \frac{(t-1)^{10}}{t^8} = \frac{(t-1)^8(2t^2-2t+1)}{t^8} \quad \ldots\ldots\ldots\ldots (3.3)$$

…and so on

As a result the following Divergence measures of Csiszar's f-divergence measure

$J_1(P,Q) = \frac{(p_i-q_i)^4(2p_i{}^2-2p_iq_i+q_i^2)}{p_i{}^4q_i}$ ……….. (3.4)

$J_2(P,Q) = \frac{(p_i-q_i)^6(2p_i{}^2-2p_iq_i+q_i^2)}{p_i{}^6q_i}$ ……..…….. (3.5)

$J_3(P,Q) = \frac{(p_i-q_i)^8(2p_i{}^2-2p_iq_i+q_i^2)}{p_i{}^8q_i}$ ……….. (3.6)

…so on.

Similarly we can generate various other series of Divergence measures using the properties of convex functions.

IV. RELATION AMONG NEW AND OTHER WELL KNOWN MEASURES & NEQUALITIES.

In this section we will drive some equations and inequalities relating$J_K^*(P,Q)$ (for the Case k=1, k=2 with the divergence measures (1.2), (1.4), (1.5) and (1.6).

Next,

From (2.8), we get

$$J_1^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i-q_i)^4}{p_i^2\, q_i} \exp\left\{\frac{(p_i-q_i)^2}{p_i^2}\right\}$$

$$= \sum_{i=1}^{n} \left[\frac{(p_i-q_i)^2}{(p_i+q_i)}\left(\frac{p_i+q_i}{p_iq_i}\right).\left(\frac{(p_i-q_i)^2}{p_i}\right)\exp\left\{\frac{(p_i-q_i)^2}{p_i^2}\right\}\right]$$

$=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{(p_i+q_i)}.2\ \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2p_iq_i}\right).\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i}\exp\left\{\frac{(p_i-q_i)^2}{p_i{}^2}\right\}$

Now, using (1.4), (1.5) and (1.6) (for K=1), we get

$J_1^*(P,Q) = \Delta(P,Q).\frac{2}{H\,(P,Q)}.R_1^*\,(P,Q)$……..(4.1)

And$J_1^*(P,Q) \geq V^2(Q,P).\frac{1}{H(P,Q)}.\ R_1^*(P,Q) \dots\dots (4.2)$

$\because \frac{1}{2}V^2(P,Q) \leq \Delta$ (P, Q)

$$J_1^*(P,Q) \leq \chi^2\,(Q,P)\,.\,\frac{2}{H(P,Q)}.\ R_1^*(P,Q) \dots .\,(4.3)$$

$\because \Delta$(P, Q) $\leq \chi^2\,(Q,P)$

$$J_1^*(P,Q) \leq [\chi_1^2(Q,P) + \chi_1^4(Q,P)].\ \frac{2}{H(P,Q)}.\ R_1^*(P,Q) \dots .\,(4.4)$$

$\because \Delta$(P, Q) $\leq \chi_1^2(Q,P) + \chi_1^4\,(Q,P)$

$$J_1^*(P,Q) \geq 4h(P,Q).\ \frac{2}{H(P,Q)}.\ R_1^*(P,Q) \dots .\,(4.5)$$

$\because 2h(P,Q) \leq \Delta$ (P, Q)

Next from(2.4),

$$J_k^c(P,Q) = \sum_{i=1}^{n} \frac{(P_i-q_i)^{2k+2}}{p_i^{2k}q_i}\ ; k = 1, 2, 3, \dots$$

For k=1

$$J_1^c(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^4}{p_i^2 q_i}$$

$$\Rightarrow \quad J_1^c(P,Q) = 2\sum_{i=1}^{n} \frac{(p_i-q_i)^2}{(p_i+q_i)} . \sum_{i=1}^{n} \left(\frac{p_i+q_i}{2p_iq_i}\right) . \sum_{i=1}^{n} \frac{(P_i-q_i)^2}{p_i}$$

Using(1.2)(1.4)and (1.5), we get

$$J_1^c(P,Q) = 2\,\Delta(P,Q).\frac{1}{H(P,Q)}\chi^2(Q,P) \ldots.(4.6)$$

And $\quad J_1^c(P,Q) \geq 4h(P,Q).\frac{1}{H(P,Q)}\chi^2(Q,P) \ldots\ldots(4.7)$

$\because 2h(P,Q) \leq \Delta$ (P, Q)

$$J_1^c(P,Q) \geq V^2(Q,P).\frac{1}{H(P,Q)}.\chi^2(Q,P) \ldots.(4.8)$$

$\because \frac{1}{2}V^2(P,Q) \leq \Delta$ (P, Q)

V. CONCLUSIONS

In this paper, we can find out different information divergence measures using properties of Csiszar'sf divergence and convex function. Divergence measures give most important key results for information theory because of we can derive different information divergence measures for different values of k. From the Figure(1), we get a steeper slope if value of $\frac{p_i}{q_i} < 0.5$ for increasing values of k .

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