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s-Degenerate Helices in 5-Dimensional Minkowski Space

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***Abstract*—In this paper, it is characterized the 2-degenerate helices and 3-degenerate helices (that is, 2 and 3-degenerate curves with constant Cartan curvatures) in 5-dimensional Minkowski space and obtained that a classification of them.**

***Index Terms*—Cartan curvatures, Cartan references, helices, s-degenerate curves.**

I. Introduction

The geometry of null hypersurfaces in spacetimes has played an important role in the development general relativity, as well as in mathematics and physics of gravitation.

General null hypersurfaces, consist of investigating the curves that live in those hypersurfaces. In this sense, the null curves in Loretzian space forms has been studied by several authors(see [3],[4],[5])

However, in a null hypersurface there are many other curves distinct from the null ones. They are spacelike curves with a null higher derivative, that is, s-degenerate curves. These curves has been studied by A. Ferrandez, G. Gimenez, and P. Lucas(see [2]).

In this paper, it is characterized the 2-degenerate helices and 3-degenerate helices (that is, 2 and 3-degenerate curves with constant Cartan curvatures) in 5-dimensional Minkowski space and obtained that a classification of them.

II. Prelaminaries

Let E be a real vector space with a symmetric bilinear mapping $g: E \times E \rightarrow \mathbb{R}$. We say that g is *degenerate* on *E* if there exists a vector $\xi \neq 0$ in *E* such that

$$g(\xi, v) = 0, \quad \text{for all } v \in E;$$

otherwise, g is said to be *non-degenerate*. The *radical* (also called *null space*) of *E*, with respect to g, is the subspace $Rad(E)$ of *E* defined by

$$Rad(E) = \{\xi \in E: g(\xi, v) = 0, v \in E;\}.$$

The dimension of $Rad(E)$ is called the *nullity degree* of g (or *E*) and is denoted by r_E.

If F is a subspace of E, then we can consider g_F the symmetric bilinear mapping on $F \times F$ obtained by restricting g and define r_F as the nullity degree of F (or g_F). For simplicity, we will use $\langle , \rangle$ instead of g or g_F.

A vector v is said to be *timelike*, *lightlike* or *spacelike* provided that $g(v,v) < 0$, $g(v,v) = 0$ $(and\ v \neq 0)$, $or\ g(v,v) > 0$, respectively. The vector $v = 0$ is assumed to be *spacelike*. A *unit vector* is a vector u such that $g(u,u) = \pm 1$.

Two vectors u and v are said to be *orthogonal*, written $u \perp v$, if $g(u,v) = 0$. Similarly,two subsets U and V of E are said to be *orthogonal* if $u \perp v$ for any $u \in U$ and $v \in V$.

Given two orthogonal subspaces F_1 and F_2 in E with $F_1 \cap F_2 = \{0\}$, the orthogonal direct sum of F_1 and F_2 will be denoted by $F_1 \perp F_2$.

Lemma 1 *Let (E,* $\langle , \rangle$*)be a bilinear space and let F be a hyperplane of* E. Let $r_F = dimRad(F)$ and $r_E = dimRad(E)$. Then the following statements hold:

i) If $r_F = 0$ and $r_E = 1$, then there exists a *null vector* L such that

$$E = F \perp span\{L\}.$$

ii) If $r_F = r_E$ and $r_E = \{0,1\}$, then there exists a *non-null* unit vector V such that

$$E = F \perp span\{V\}$$

Moreover, if $Rad(E) = \{0\}$ then V is unique, up to the sign.

iii) If $r_F = 1$ and $r_E = 0$, and $F = F_1 \perp L$, where $L \in Rad(F)$and F_1 is non-degenerate, then there exists a unique null vector N such that $\langle L, N \rangle = \varepsilon, \varepsilon = \pm 1$, and

$$E = span\{L\} \oplus span\{N\}) \perp F_1$$

III. s-Degenerate Curves

Let (M_1^n, ∇) be an oriented Lorentzian manifold and let $\gamma: I \longrightarrow M_1^n$ be a differentiable curve in M_1^n. For any vector field V along γ, let V' be the covariant derivation of V along γ. Write $E_i(t) = span\{\gamma'(t), \gamma''(t), \dots, \gamma^{(i)}(t)\}$, where $t \in I$ and $i = 1,2,\dots,n$. Let *d* be the number defined by $d = max\{i: dim(E_i(t)) = i\ for\ all\ t\}$.

Definition 2 With the above notations, the curve $\gamma: I \longrightarrow M_1^n$ is said to be an *s-degenerate* (or *s-lightlike*) *curve* if for all $1 \leq i \leq d$, $\dim\ (Rad(E_i(t))$ is constant for all *t*, and there exists *s*, $0 < s < d$, such that $Rad(E_s) \neq \{0\}$ and $Rad(E_j) = \{0\}$ for all *j*<*s*.

Theorem 3 Let $\gamma: I \longrightarrow \mathcal{M}_1^n, n = m + 2$, be an s-degenerate unit curve, $s > 1$, and suppose that

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$\{\gamma'(t), \gamma''(t), \dots, \gamma^{n}(t)\}$ spans $T_{\gamma(t)}\mathcal{M}_1^n$, for all t. Then there exists a unique Frenet frame satisfying equations:

$$\gamma' = W_1$$
$$W_1' = k_1 W_2$$
$$W_i' = -k_{i-1}W_{i-1} + k_i W_{i+2}, 2 \le i \le s-2$$
$$W_{s-1}' = -k_{s-2}W_{s-2} + L$$
$$L' = k_{s-1}W_s$$
$$W_s' = \varepsilon k_s L - \varepsilon k_{s-1} N$$
$$N' = -\varepsilon W_{s-1} - k_s W_s + k_{s+1}W_{s+1}$$
$$W'_{s+1} = -\varepsilon k_{s+1} L + k_{s+2} W_{s+2}$$
$$W_j' = -k_j W_{j-1} + k_{j+1}W_{j+1}, s+2 \le j \le m-1$$
$$W'_m = -k_m W_{m-1}$$

for certain functions $\{k_1, k_2, \dots, k_m\}$.

Definition 4 An *s-degenerate curve*, $s > 1$, satisfying the above conditions is said to be an s-degenerate Cartan curve. The reference and curvature functions given by Theorem 3 will be called the *Cartan reference and Cartan curvatures* of γ, respectively.

Observe that when $m>s$ then $\varepsilon = -1$ and $k_i > 0$ for $i \neq s$, and $k_m > 0$ ($k_m < 0, resp.$) according to $\{\gamma', \gamma'', \dots, \gamma^{(n)}\}$ is positively or negatively oriented, respectively. However, when $m=s$ then $\varepsilon = -1$ or $\varepsilon = 1$ according to $\{\gamma', \gamma'', \dots, \gamma^{(n)}\}$ is positively or negatively oriented, respectively, and $k_i > 0$ for $i \neq s$.

Definition 5 An *s-degenate helix* in M_1^n is an s-degenerate Cartan curve having constant Cartan curvatures.

Theorem 6 Let $\gamma: I \longrightarrow \mathbb{R}_1^n$ be an s-degenerate Cartan curve and D_t denote the covariant derivative in $\mathbb{R}_1^n$ along γ. Then for any vector field V along γ we have $D_t V = V'$, where $\langle , \rangle$ denotes the standart metric in $\mathbb{R}_1^n$. If $\{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_m\}$ is the Cartan reference, then equations write down as follows:

$$\gamma' = W_1,$$
$$W'_1 = k_1 W_2,$$
$$W'_i = -k_{i-1}W_{i-1} + k_i W_{i+2}, 2 \le i \le s-2,$$
$$W'_{s-1} = -k_{s-2}W_{s-2} + L,$$
$$L' = k_{s-1}W_s,$$
$$W'_s = \varepsilon k_s 1 - \varepsilon k_{s-1} N,$$
$$N' = -\varepsilon W_{s-1} - k_s W_s + k_{s+1}W_{s+1},$$
$$W'_{s+1} = -\varepsilon k_{s+1} L + k_{s+2}W_{s+2},$$
$$W'_j = -k_j W_{j-1} + k_{j+1}W_{j+1}, s+2 \le j \le m-1,$$
$$W'_m = -k_m W_{m-1}.$$

IV. S-DEGENERATE HELICES IN R_1^5

Proposition 7 Let $\gamma: I \to \mathbb{R}_1^5$ be an 2-degenerate Cartan curve. If γ is a helix, γ satisfies following differential equation:

$$\gamma^{(6)} = 2\varepsilon\sigma_1\sigma_2\gamma^{(4)} + \varepsilon\sigma_1\sigma_3^2\gamma''' + \sigma_1^2\gamma''$$

Proof: By Theorem 6, we have

$$\gamma' = W_1 \quad (1)$$
$$W_1' = L \quad (2)$$
$$L' = \sigma_1 W_2 \quad (3)$$
$$W_2' = \varepsilon\sigma_2 L - \varepsilon\sigma_1 N \quad (4)$$
$$N' = -\varepsilon W_1 - \sigma_2 W_2 + \sigma_3 W_3 \quad (5)$$
$$W_3' = -\sigma_3 W_2. \quad (6)$$

Cartan curvatures of γ are constant, since γ is a helix curve. By differentiating both sides of the Eq. (1), and using Eq.(2) and Eq. (3), we have

$$\gamma'' = L$$

and

$$\gamma''' = \sigma_1 W_2. \quad (7)$$

By taking derivatives of this equation, and using Eq. (4), Eq. (5) and Eq. (6) we obtain the following differential equation

$$\gamma^{(6)} = 2\varepsilon\sigma_1\sigma_2\gamma^{(4)} + \varepsilon\sigma_1\sigma_3^2\gamma''' + \sigma_1^2\gamma''$$

Proposition 8 Let $\gamma: I \to \mathbb{R}_1^5$ be an 3-degenerate Cartan curve. If γ is a helix, γ satisfies following differential equation:

$$\gamma^{(6)} - (2\varepsilon\sigma_2\sigma_3 - \sigma_1^2)\gamma^{(4)} - (2\varepsilon\sigma_1^2\sigma_2\sigma_3 + \sigma_2^2)\gamma'' = 0$$

Proof: By Theorem 6, we have

$$\gamma' = W_1$$
$$W_1' = L$$
$$L' = \sigma_1 W_2$$
$$W_2' = \varepsilon\sigma_2 L - \varepsilon\sigma_1 N$$
$$N' = -\varepsilon W_1 - \sigma_2 W_2 + \sigma_3 W_3$$
$$W_3' = -\sigma_3 W_2.$$

Then, by a straightforward computations, the result is obtained as in the the proof of the Proposition 7.

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