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On Computation of Polynomial Knot Invariant

Owino B., Mueni. M.

Abstract— **In this paper we present a combinatorial way of representing states that arise from the various smoothings of knots. We also apply such presentations in calculating polynomial invariant of knots.**

Index Terms— **Kauffman bracket polynomial, Knot Invariants, Oriented links, Smoothing, States.**

I. Introduction

Knots have been in existence since time immemorial and still and have been an object of study by many researchers.Mathematical study of knots can be traced back to the work of Alexander and Reidemeister [2, 4]. More recently, knots have found application in other disciplines including Biology, where they have been used to study DNA and RNA [9, 1]. Knots have also been applied in chemistry and physics as well [3, 8, 7].

Given two knots it is a difficult problem to distinguish between them or tell if they are equivalent. Several knot invariants have been developed over the years for this purpose. Polynomial invariants such as Alexander polynomial [2] , Jones polynomial [11], Conway polynomial [5], HOMFLY polynomial [10] and the Kauffman bracket polynomial [6] have been used in distinguishing between knots. The main problem with topological invariant of a knot is the fact that computations increases tremendously whenever one moves to a knot with more crossings. In the case of Kauffman bracket polynomial for instance, the number of states doubles with each additional crossing. Also, smoothing about the crossings result into closed curves, which may be quite complex to recognize when the number of crossings is large.

We shall show how to count and put together the states arising from smoothing in such a way that states with same Kaufmann polynomials. With this count, one can then calculate the Kauffman bracket polynomial for a given knot with n crossings without drawing all the 2^n states. This significantly reduces the complexity in such computations.

The outline of the paper is as follows. In section 2 we introduce terminologies and build notations used in the paper. In section 3 we present the combinatorial argument and show how it applies to presentation of the states before drawing conclusions in section 4.

Owino B., Department of Pure and applied Mathematics. Technical University of Kenya.
Mueni. M., Department of Pure and applied Mathematics. Technical University of Kenya,

II. KAUFFMAN BRACKET POLYNOMIAL

A permutation is the different arrangements of a given number of elements taken one by one, some or all at the same time where the order is important. A combination is a selection of some or all of a number of different objects where the order is not important.

Given a knot diagram D, then for a fixed crossing, one can rotate the knot diagram so that the crossing looks like the one shown below.

There are two ways of resolving this crossing. One can connect the 'northwest' strand to the 'southwest' strand and the 'northeast' strand to the 'southeast' strand. This is a resolution of type A also known as type A smoothing. Another resolution is achieved by connecting 'northwest' strand to the 'northeast' strand and connecting the 'southwest' strand to the 'southeast' strand. This is a resolution of type B or a type B smoothing.

The Kauffman bracket is a Laurent polynomial as it allows both positive and negative integer powers. It is defined in terms of states and the number of type A and type B smoothings. The state of a diagram D of a knot with type A and type B crossings is a diagram that results after applying the smoothings to each crossing of D. For each state S, $<D|S>=t^{a-b}$ where a is the number of type A smoothings and b is the number of type B smoothings.

Then the Kauffman bracket $<D>$ of the diagram D is obtained by taking the sum of all the terms where $|s|$ denotes the number circles in the state S.

A knot is called oriented if a direction is assigned to it. A crossing is called right-handed if an observer stationed on the over passing arc and facing in the direction of that arc observes the under passing arc's direction as from right to left. Otherwise, we will call the crossing left-handed.

The Kauffman bracket polynomial of an oriented knot bracket is defined as $(-t)^{3(r-l)} <D>$, where r denotes the number of right-handed crossings of a knot diagram D, l the number of left-handed crossings of D and $<D>$ denotes the Kauffman bracket.

Since a knot with n crossings has 2^n states, calculation of $<D>$ involve taking a sum of 2^n polynomials. However, these states can be grouped so that one works with the sum over the permutations arising for some fixed a and b.

III. COMBINATORIAL PRESENTATION OF STATES.

We now present the combinatorial representation of the states. We shall call two circles adjacent if they are related by a smoothing. Circles will be labeled with upper case letters. Also crossings on the original knot will be numbered in some order, and consistency of this order maintained. Since there are only two types of smoothing, each state shall bear symbols after the smoothing type and an adjacency relation. For example, the symbol

$$ {}^{2}_{1}A_3^1A_2^3A \qquad (1)$$

stands for the state arising from applying three type A smoothing applied to the three crossings of the trefoil knot.

Rather than calculating the Kauffman bracket for individual state, we consider classes of cases arising by fixing A and B. This then allows for a direct calculation of number of circles.

The first step in the process, after labeling the crossings, is to applying type A smoothing only or type B smoothing only and draw the resulting state. For our purpose, we start by considering only type A smoothing only. The next step is to apply type B smoothing, changing the symbols one at a time until we end up with only type B smoothing for each type A smoothing in the initial state. Throughout the process, we record the possible number of states grouped according to the number of circles that arise. The following two tables give results for left-handed Trefoil knot.

Number of smoothing type		Number of circles	number states
a	b		
3	0	3	1
2	1	2	3
1	2	1	3
0	3	2	1

Table 1: States for Trefoil Knot

We now use the figure-8 Knot to illustrate the argument. Consider the figure-eight knot in "Fig 1" below.

Fig 1: Figure 8 Knot.

Let us number the crossings from top, middle, bottom right then bottom left as 1, 2, 3 and 4 respectively. We will choose our orientation such that we move downwards from 1 to 3. After carrying out type A smoothing about all these crossings, we end up with three circles, one circle on top of the other and a third circle enclosing the two, which we may label as A, B, and C. For our case, let A be the circle enclosing B and C where B is on top of C. In terms of notation (1), this initial state can be expressed a, ${}^{1}_{2}A_3^2A_1^3A_1^3A$.

Number of smoothing of type		Number of circles	number states	Calculation for number of states
a	b			
4	0	3	1	1
3	1	2	4	$2({}^2C_1 \times {}^2C_1)$
2	2	1	5	${}^2C_2 \times {}^2C_0 + 2({}^2C_1 \times {}^2C_1)$
		3	1	${}^2C_2 \times {}^2C_0$
1	3	2	4	$2({}^2C_2 \times {}^2C_1)$
0	4	3	1	${}^4C_4$

Table 2: states for Trefoil Knot

Remark.

Note that the number of circles for the first case can be determined from the drawing, while those for the second case are one less. A similar argument is true for the last case and the second last case, that is, they can easily be obtained diagrammatically.

With this table in place, the Kauffman bracket polynomial can be calculated as follows;

$$F_4 = (-t)^{3(r-l)}\left(\sum < D\,|\,S > (-t^{-2}-t^2)^{|s|-1}\right)$$

$$= (-t)^{3(2-2)}\begin{pmatrix} t^4\left(-t^{-2}-t^2\right)^2 + 4t^2\left(-t^{-2}-t^2\right) + \\ 5t^0\left(-t^{-2}-t^2\right)^0 + t^0\left(-t^{-2}-t^2\right)^2 + \\ 4t^{-2}\left(-t^{-2}-t^2\right) + t^{-4}\left(-t^{-2}-t^2\right)^2 \end{pmatrix}$$

$$= t^{-8} - t^{-4} + 1 - t^4 + t^8$$

IV. CONCLUSION

Calculation of Kauffman bracket polynomial for a knot with n crossings involve calculation 2^n states. For the figure 8 knot, one needs to calculate 16 such polynomials. Grouping the states by smoothing type as well as the number of circles has given rise to only 6 classes. In general, for "nice" instances, the number of groupings by smoothing types and number of circles will give n+1 cases. Such a grouping makes calculations less tedious and also minimizes the errors that may arise while dealing with many cases.

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Ben Owino.
Department of Pure and applied Mathematics.
Technical University of Kenya.

Martha Mueni.
Department of Pure and applied Mathematics.
Technical University of Kenya,