World Journal of Research and Review (WJRR)
ISSN: 2455-3956, Volume-6, Issue-5, May 2018 Pages 65-72

Nonlinear Stability of Roll Waves Down an Inclined Falling Film

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***Abstract*— The present work is concerned with surface instabilities of non-Newtonian liquid films, usually called roll waves (RW). A thin liquid film in which the shear stress is modeled as a power-law is considered to study the stability of nonlinear roll waves down inclined plane walls. In the long wave approximation, depth-integrated continuity and momentum equations are derived by applying Karman's momentum integral method. As the linearized instability analysis of uniform flow only provides a diagnosis of instability, the modulation equations for wave series are derived and a stability criterion depending on two parameters (integro-differential expression) is obtained. The main difficulty to establish the stability domain is due of the presence of singularities near infinitesimal and maximal amplitudes. Numerical calculations are performed using asymptotic formulas near the singularities. The stability diagrams are presented for some values of the flow parameters. They reveal that there are situations wherein at critical values of the flow parameters, where the waves disappear. For the prediction and control of the free-surface profile, it is one of the main reasons for carrying out research in this area, as RW are generally an undesirable phenomenon.**

***Index Terms*— Power-Law Fluid, Inclined Plane Wall, Modulation Equations, Nonlinear Stability of Roll Waves**

I. Introduction

It is well known that a steady discharge in inclined long open channels may develop into roll waves. In the initial stage of their development the waves are of small amplitude and the free surface is smooth.

A steady shallow turbulent flow becomes unstable if certain criteria are satisfied and may evolve towards breaking bores propagating at almost constant speed and followed by smooth long waves. Such waves have been described for the first time by Cornish [1] who has observed them in water runways. The quantitative analysis of turbulent RW of finite amplitude by assuming the Chézy law for bottom resistance has been first given by Dressler [2]. In Dressler's theory the length scale is a free parameter. However, it is well known that these waves develop rather in a narrow band of dimensionless wave numbers [3]. Hence for, the question on stability of nonlinear RW arises.

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In the thin laminar viscous liquid flow down an inclined plane wall is of considerable importance for wide applications in the fields of chemical technology [4][5][6][7][8]. Over a range of flow parameters, a variety of free instabilities may occur. Roll waves are among these flow patterns, which may exhibit quasi-periodic spatial structures of the free surface. A description of nonlinear analysis of RW has been considered in Liapidevskii [9]; Boudlal & Liapidevskii [10] respectively in two-dimensional flow and for arbitrary cross sections by deriving the modulation equations and stability diagrams of periodic series of nonlinear RW [11][12][13]. Laminar sheet flow with quadratic distribution of velocity profile has been studied by Alekseenko & Nakoriakov [5]; Buchin & Shaposhnikova [14]; Julien & Hartley [15].

From the experimental work on Newtonian thin film of Kapitza [16], Liu, Paul & Gollub[17], among others, have experimentally investigated the linear stability. A detailed study of this work can be found in the book of François Charru [18]. Due principally to the effect of inertia, surface instabilities of flows in Newtonian and non Newtonian shallow layer fluids down inclines have been observed for a long time. Over a range of the flow rates several kinds of waves may occur. These flows play an important role in natural and various industrial processes. During the fourth decades of the 20th century up to now, investigation of RW principally instigated by industry needs has been intensive. Generally, the generation and amplification of surface instabilities is an undesirable phenomenon because of destructive damage which may be induced.

A report for the most work conducted in this area can be found in extensive references quoted by Ng & Mei [4]. The flow of non-Newtonian power law fluids occurs frequently in the fields of chemical technology and geophysical process [4][6][19][20]. The waves may develop into finite-amplitude permanent wave trains accompanying a series of jumps. RW have been observed and reported by several authors in mud flow [4] and extended to granular flows in dense regime [21]. An extensive treatment has been conducted in this turbulent regime area by Dressler [2]; Boudlal & Liapidevskii [11]; Jeffreys [22]; Femndez-Nieto, Nobel & Vila [23], among others. Dressler's theory, originally developed for fully turbulent flows, has been extended to Newtonian flow by Ishihara [24]; Iwasa [25] and to non-Newtonian thin film in Ng & Mei [4]. It has been shown, in particular, that for highly non-Newtonian fluids, very long waves may occur even if the corresponding flow is instable according to the linear theory. On the other hand, much amplitude RW are inadmissible relative to the loss of energy across the shocks. So, the on stability of nonlinear RW arises once more.

In this theory the length of roll waves is the free parameter, and the flow characteristics of saturated waves, which stop to grow after attaining some critical amplitude, cannot be predicted. Therefore, the question on stability of travelling waves of finite amplitude arises. Dissipative shallow water models let construct smooth periodic solutions describing the roll wave phenomenon, nevertheless in these models the length of periodic waves is a free parameter again.

The aim of the paper is to give a nonlinear study on stability of permanent roll waves on a shear thinning fluid in the frame of one-dimensional, unsteady, gradually varied, laminar mud flow with the shear stress being evaluated in a conventional manner. Starting from long wave's equations averaged over the normal to the bed, on the basis of C. O. Ng and C.C. Mei's model, periodic roll waves are constructed. As the amplitude and the phase velocity of waves are slowly varying during their propagation and as these variations give rise to instability, this problem of stability can be solved by deriving modulation equations for wave packets [11][12][26][27]. The stability criterion for nonlinear roll waves is formulated in terms of hyperbolicity of modulation equations that need calculation of averaged quantities. The main difficulty to establish the stability domain on a roll wave diagram is due to the singularities in the hyperbolicity condition of modulation equations for the waves of the infinitesimal and maximal amplitude. Using an asymptotic analysis, the stability conditions of roll waves of maximal amplitude, as well as the approximate position of boundaries of the hyperbolicity domain are obtained.

Boudlal & Liapidevskii [11] have already given a non linear study of stability of permanent roll waves on shear thinning fluid on vertically falling films in the basis of one-dimensional, unsteady gradually varied, laminar mud flow with shear stress being evaluated in conventional manner. Note that for a vertically falling film, the system being self-similar, the modulation equations take a rather simple form and the hyperbolicity criterion is reduced to a condition for a function of one variable. Here we present an inclined flow. The stability condition depends on two governing parameters. Numerical calculations of stability diagrams corresponding to an inclined plane wall are presented. All results presented herein can be regarded as a generalization to a power law mud fluid in laminar flow regime of non linear stability method already applied to Newtonian turbulent flows in open channels [10].We find that roll waves can disappear for a critical value of the flow parameters, it is very important in surface coating processes: paints and varnishes, inking in the printing press, depositing thin layers on magnetic tapes and photographic films, etc. In these situations, an essential quality of the liquid films is the uniformity of their thickness, and the roll waves must be absolutely avoided.

II. GOVERNING EQUATION

We consider a two-dimensional flow of a thin fluid film down a plane inclined with an angle θ, $(0<\theta<\pi/2)$. The coordinate axes are chosen such that the $Ox-axis$ points in the flow direction, and $Oy-axis$ perpendicular to it, directed upward (Fig.1).

In the long wave approximation, the film thickness depends only on x and t, and the pressure is hydrostatic: $p=\rho g_n(h-y)$, where $g_n = g\cos\theta$. The shear is modeled by a power law of the form [4][11]:

$$\sigma = \mu_n \left(\frac{\partial u}{\partial y}\right)^n \quad (0<n\le 1) \qquad (1)$$

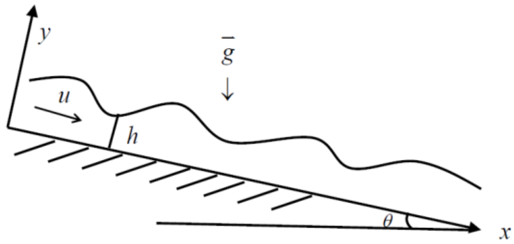


Fig.1 Sketch of flow model

With the following dimensionless quantities defined by:

$$t=\frac{l_0}{u_0}t^*,\ x=l_0x^*,\ \bar{u}=\bar{u}_0\bar{u}^*,\ (h,y)=h_0(h^*,y^*),$$

$$\sigma_b=\sigma_{b0}\sigma_b^*,\ l_0=\frac{\bar{u}_0^2}{g\sin\theta},\ \sigma_{b0}=\mu_n\left[\left(\frac{1+2n}{n}\right)\frac{\bar{u}_0}{h_0}\right]^n, \qquad (2)$$

$$\sigma_b^*=\left(\frac{\bar{u}^*}{h^*}\right)^n,\ \alpha=\frac{g_n h_0}{\bar{u}_0^2}$$

Here h is the depth, $\bar{u}$ is the mean velocity, t is the time, g is the gravity acceleration and σ_b is the bottom stress, the subscript 0 stands for the reference quantities. The viscosity coefficient is denoted by μ_n with dimension $\left[ML^{-1}T^{n-2}\right]$ and n is the flow index $(0<n\le 1)$. The case $n=1$ corresponds to the Newtonian fluid and μ_1 is the ordinary dynamic viscosity [4][11].

In dimensionless variables, the governing equations of mass and momentum conservation, averaged in the ordinate direction may take the form (the asterisks are omitted for simplicity) [1]:

$$\begin{cases} h_t+(\bar{u}h)_x=0, & (3a)\\ (\bar{u}h)_t+\left(\beta\bar{u}^2h+\frac{1}{2}\alpha h^2\right)_x=h-\left(\frac{\bar{u}}{h}\right)^n & (3b)\end{cases}$$

The main objective of this work is to investigate the nonlinear stability of roll waves. For this purpose the approach developed in [11] for RW on vertical plane will be applied for thin viscous film flows on an incline.

III. WITHAM'S STABILITY APPROACH

Let a small perturbation of subscript 1 be added to the base uniform flow of subscript 0 as follows:

$$h=h_0+h_1,\ \bar{u}=\bar{u}_0+\bar{u}_1\ (h_1\ll h_0,\ \bar{u}_1\ll\bar{u}_0) \qquad (4)$$

From a standard linear analysis it is shown in [1] that the stationary solution of (3):

$$\bar{u}_0=h_0^{\frac{1+n}{n}},\ h_0=\text{cte} \qquad (5)$$

is unstable if the following condition is satisfied [1] :

$$\alpha<\frac{1+2n}{n^2} \qquad (6)$$

The same criterion (6) of transition to instable flow can be derived by the method promoted by Witham's method [12]. Indeed, after eliminating $\bar{u}_1$ and derivation, the linearized system (3) takes the following condensed form:

$$\left[\frac{\partial}{\partial t}+\lambda_d^+\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t}+\lambda_d^-\frac{\partial}{\partial x}\right]h_1+\left\{n\left(\frac{\partial}{\partial t}+\lambda_c\frac{\partial}{\partial x}\right)\right\}h_1=0 \quad (7)$$

There is stability if the velocity of the kinematic waves lies between the velocities of the dynamic wave $\lambda_d^{\pm}$, i.e.

$$\lambda_d^- < \lambda_c < \lambda_d^+, \qquad (8)$$

If $\bar{u}_0 = h_0 = 1$, we obtain:

$$\lambda_c = 2+\frac{1}{n},\ \lambda_d^{\pm} = \beta \pm \sqrt{(\beta-1)\beta+\alpha} \qquad (9)$$

On substituting from (8), the condition (6) is recovered.

IV. ROLL WAVES

We intend to construct discontinuous periodic wave solutions which propagate with a constant speed D $(0<\bar{u}<D)$ (Fig.2). In the frame of reference accompanying the waves the flow is steady and equations (3) may be expressed in terms of the single variable $\xi = x - Dt$. With this transformation, equation (3a) may be integrated directly to give

$$q=(D-\bar{u})h \qquad (10)$$

where the constant q is equal to the apparent discharge rate or progressive discharge according to [4].

Between two successive discontinuities the momentum equation is valid, making use of (10), equation (3b) leads to an ODE for h which may read:

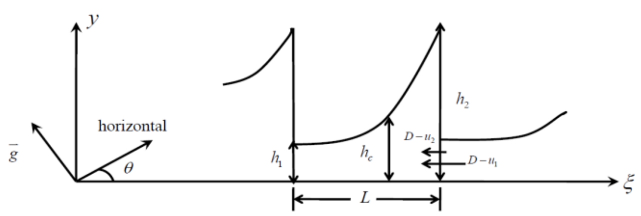


Fig.2 Sketch in the frame of reference accompanying the roll waves

$$\frac{dG}{d\xi}=\Delta\frac{dh}{d\xi}=F \qquad (11)$$

with:

$$G(h) = -D(\bar{u}h)+\beta\bar{u}^2h+\alpha\frac{h^2}{2} \qquad (12)$$

$$\Delta(h)=(\beta-1)D^2-\beta\frac{q^2}{h^2}+\alpha h \qquad (13)$$

$$F(h)=h-\left(\frac{D}{h}-\frac{q}{h^2}\right)^n \qquad (14)$$

The profiles of $G(h)$ and $\Delta(h)$ are shown in Fig.3. As it can be seen $G(h)$ must present a minimum which will be called the critical depth and denoted by h_c.

To complete the construction of roll wave, we must give the jump conditions. From the system (3), with the help of (10), the relations at jump discontinuity are reduced to

$$\begin{cases} G(h_1)=G(h_2) \\ q(h_1)=q(h_2)=q(h_c) \end{cases} \qquad (15)$$

where subscripts 1 and 2 denote the right and the left sides of the discontinuity. Equations (15) show that the roll wave depend on two parameters, for example: h_1, h_c. From system (15), we get a relation between the two depths h_1 and h_2 as:

$$h_2=\frac{4\beta q^2}{\left(\alpha h_1^2+2(\beta-1)D^2h_1\right)+\sqrt{(\alpha h_1^2+(\beta-1)2D^2h_1)^2+8\alpha\beta q^2h_1}} \qquad (16)$$

which is reduced to:

$$h_2=\frac{\beta q^2}{(\beta-1)D^2h_1} \qquad (17)$$

for $\alpha=0$ (vertical wall).

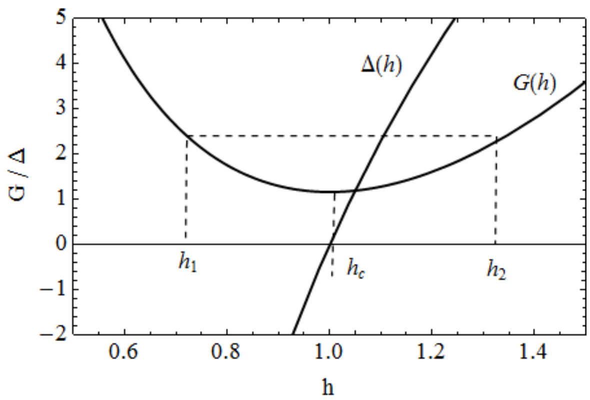


Fig.3 Profile of G and Δ vs. h

For standard roll waves with the only one jump on the period that divides the monotone smooth parts of flow. Therefore, it is necessary for the roll wave existence that the subcritical flow behind the jump $(\Delta>0)$ transforms into the supercritical flow $(\Delta<0)$ before the next jump, and there exists the critical depth h_c on the period. For the roll wave existence it is necessary that $F(h)$ and $\Delta(h)$ vanish at the critical depth h_c simultaneously (Fig.4), i.e.

$$F(h_c)=\Delta(h_c)=0 \qquad (18)$$

After some algebra manipulations, taking (10) into account, the system (18) leads to

$$\begin{aligned} &q=h_c^{(2n+1)/n}(\phi-1),\ D=h_c^{(n+1)/n}\phi, \\ &\phi=\beta+\left\{(\beta-1)\beta+\alpha h_c^{-(n+2)/n}\right\}^{1/2} \end{aligned} \qquad (19)$$

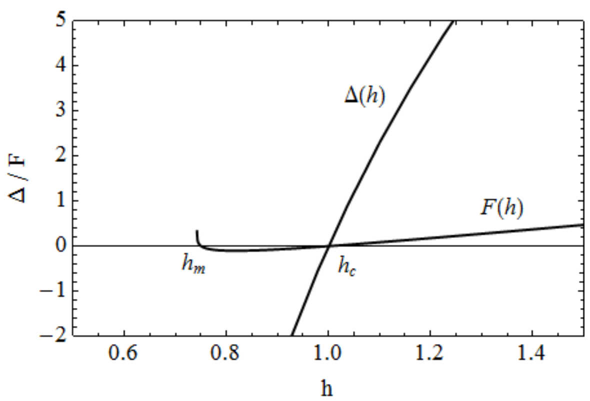


Fig.4 Profile of Δ and F vs. h

For h_1 and h_c given, the conditions:

$$\begin{cases} F(h) > 0 & \text{for} \quad h_c < h < h_2 \\ F(h) < 0 & \text{for} \quad h_1 < h < h_c \end{cases} \qquad (20)$$

are necessary and sufficient for the formation of roll waves. It is shown in [1] that there exist solutions for roll waves only if $h_m < h_1 < h_c$ where h_m and h_c are the two roots of the equation $F(h) = 0$, as shown in Fig.4.

According to the reported observations, between two successive jumps the surface profile must increase, otherwise the slope $dh/d\xi$ must be positive; it is also the condition of irreversibility of hydraulic jump.

Moreover, as $\Delta'(h_c)$ is positive, the necessary condition

$$F'(h_c) > 0 \qquad (21)$$

is required, i.e.

$$\alpha < \left(\frac{1+2n}{n^2}\right) h_c^{\frac{n+2}{n}} \qquad (22)$$

Compared to the previous stability condition (6), we obtain two conditions of existence of roll waves: For $h_c \leq 1$, roll waves can only occur if the uniform base flow is unstable, in this case, the required linear stability criterion (6) is unconditionally satisfied, since the transition from uniform flow to intermittent flow regime is usually tackled by resorting to stability theory according to [1]. Whereas for $h_c > 1$, roll waves can occur even if the uniform flow is stable to small disturbances.

These two conditions are illustrated in Fig.5 for $n = 0.4$.

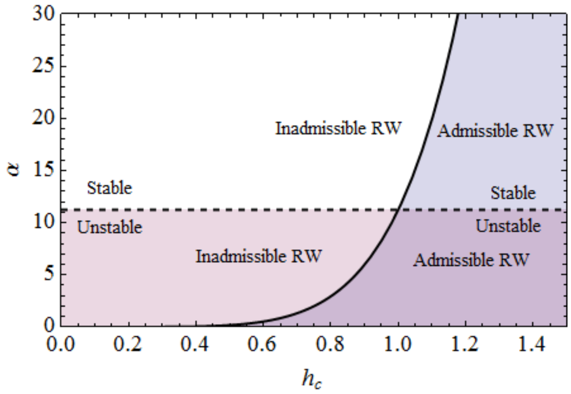


Fig.5 Regions of admissible RW: ___ $\alpha = \left(\frac{1+2n}{n^2}\right) h_c^{\frac{n+2}{n}}$ ---- stability to infinitesimal disturbances.

V. MODULATION EQUATIONS

The periodic solution of (3) is defined by two parameters, we can choose h_1 and h_c as such parameters. The problem on nonlinear stability of periodic wave trains with slowly varying values h_1 and h_c can be solved by analysis of hyperbolicity of the modulation equations for such waves. After averaging (3) over the fixed length scale, which is large enough, compared with the length of roll waves, the following modulation equations:

$$\begin{cases} \bar{h}_t + \left(\overline{uh}\right)_x = 0 \\ \left(\overline{uh}\right)_t + \left(\overline{\beta u^2 h + \frac{1}{2}\alpha h^2}\right)_x = 0 \end{cases} \qquad (23)$$

are obtained. All averaged quantities can be expressed in functions of h_1 and h_c as following:

$$\begin{aligned} & L = \int_0^L d\xi = \int_z^{w(z,h_c)} A(s,h_c)ds; \\ & \bar{h} = L^{-1}\int_z^{w(z,h_c)} sA(s,h_c)ds; \\ & \bar{B} = L^{-1}\int_z^{w(z,h_c)} B(s,h_c)A(s,h_c)ds; \\ & B(s,h_c) = \frac{\beta q^2}{s} + \frac{\alpha s^2}{2}; \\ & A(s,h_c) = \left\{(\beta-1)D^2 - \beta\frac{q^2}{s^2} + \alpha s\right\}\left\{s - \left(\frac{D}{s} - \frac{q}{s^2}\right)^n\right\}^{-1}; \\ & \overline{hu} = D\bar{h} - q; \\ & \overline{\beta u^2 h + \frac{1}{2}\alpha h^2} = \beta(D^2\bar{h} - 2qD) + \bar{B}; \\ & s \in [z, w],\ z = h_1,\ w = h_2. \end{aligned} \qquad (24)$$

In view of (24), the modulation equations take the form:

$$\begin{cases} \bar{h}_t + \left(D\bar{h} - q\right)_x = 0 \\ \left(D\bar{h} - q\right)_t + \left(\beta(D^2\bar{h} - 2qD) + \bar{B}(z,h_c)\right)_x = 0 \end{cases} \qquad (25)$$

The non stationary evolution of the governing parameters (z, h_c) for a periodic wave train is described by equation (25). We say the roll waves are stable if the modulation equations (25) for corresponding values (z, h_c) are hyperbolic. The investigation of hyperbolicity of the modulation equations can be performed more easily for the variables $\bar{h}$ and h_c. It can be done by the transformation $\bar{h} = \bar{h}(z, h_c)$ and $\tilde{B}(\bar{h}, h_c) = \bar{B}(z, h_c)$. The modulation equations take the following form:

$$\begin{cases} \bar{h}_t + \left(D\bar{h} - q\right)_x = 0 \\ \left(D\bar{h} - q\right)_t + \left(\beta(D^2\bar{h} - 2qD) + \tilde{B}(\bar{h},h_c)\right)_x = 0 \end{cases} \qquad (26)$$

The characteristics of (26) are

$$\begin{aligned} & \tau^{\pm} = \left(\frac{dx}{dt}\right)^{\pm} = \frac{\left\{2\beta D + (-2q\beta D' + \tilde{B}_{h_c})\delta^{-1}\right\} \pm \sqrt{Disc}}{2} \\ & Disc = \left\{2(\beta-1)D + (-2q\beta D' + \tilde{B}_{h_c})\delta^{-1}\right\}^2 + 4\left((\beta-1)D^2 + \tilde{B}_{\bar{h}}\right) \\ & \delta = D'\bar{h} - q' \end{aligned} \qquad (27)$$

Here the “prime” denotes the full derivation on the variable h_c. The hyperbolicity condition for (27) is

$$Disc > 0 \qquad (28)$$

It follows from (27) that the stability of RW depends on z, h_c. Moreover, there exists a value $h_m < z^* < h_c$ such that the system (26) is elliptic for $z^* < z < h_c$ and it is hyperbolic for $h_m < z < z^*$. It means that periodic travelling waves become stable only if they have the amplitude exceeding some critical value. The calculations show that the values z^*, h_m are very close to each other. Therefore, to find the stability domain on the $(z,\ h_c)$ – plane , we have to resolve the singularities in (24) at near maximal amplitude for $L \to \infty,\ \bar{h} \to h_m$. The asymptotic analysis for such case is performed in the following section.

VI. STABILITY OF ROLL WAVES OF NEAR MAXIMAL AMPLITUDE

Suppose that roll waves are defined for every critical depth h_c from an interval. It means that there are the smooth functions $z^* = z^*(h_c)$ and $w^* = w(z^*, h_c)$, and the conditions (20) are satisfied for conjugate depths z, w with $G(z) = G(w)$, $z^* < z < h_c < w < w^*$, $F(z^*, h_c) = 0$, $F_z(z^*, h_c) \neq 0,\ F(w^*, h_c) \neq 0.$

When $z \to z^*$ we can use the asymptotic formulae

$$A(s,h_c) = \frac{b(s,h_c)}{s-z^*}, \quad b(z^*,h_c) \neq 0,$$

$$L(z,h_c) = \int_z^w \frac{b(s,h_c) - b(z^*,h_c)}{s-z^*} ds + b(z^*,h_c) \ln \frac{w-z^*}{z-z^*}\bigg|_{z\to z^*} \to \infty, \tag{29}$$

$$\bar{h}(z,h_c) = \frac{1}{L}\int_z^w (s-z^*)A(s,h_c)ds + z^*\bigg|_{z\to z^*} \to z^*,$$

$$\bar{B}(z,h_c) = \frac{1}{L}\int_z^w (B(s,h_c) - B(z^*,h_c))A(s,h_c)ds + B(z^*,h_c)\bigg|_{z\to z^*} \to B(z^*,h_c),$$

Excluding the wave length L from (29) we have:

$$\bar{B}(z,h_c) = B(z^*,h_c) + \frac{(\bar{h}-z^*)}{\int_z^w b(s,h_c)ds} \int_z^w \frac{\left(B(s,h_c) - B(z^*,h_c)\right)b(s,h_c)ds}{s-z^*} \tag{30}$$

The approximate expression for the function $\widetilde{B}(\bar{h},h_c) = \bar{B}(z,h_c)$ is given by the following formulae, in which the limits of integrals in (30) are used for $z \to z^*$.

$$\widetilde{B}^*(\bar{h},h_c) = B(z^*,h_c) + B^{\prime *}(\bar{h}-z^*)$$

$$\text{with: } B^{\prime *} = \frac{\displaystyle\int_{z^*}^{w^*} \frac{(B(s,h_c) - B(z^*,h_c))b(s,h_c)ds}{s-z^*}}{\displaystyle\int_{z^*}^{w^*} b(s,h_c)ds} \tag{31}$$

The approximation (31) can be applied for calculations of the hyperbolicity domain of the modulation equations (26). For that we replace the function $\widetilde{B}(\bar{h},h_c)$ by $\widetilde{B}^*(\bar{h},h_c)$. The criterion of the hyperbolicity takes the form:

$$Disc^* = \left\{2(\beta-1)D + (-2m\beta D' + \widetilde{B}^*_{h_c})\delta^{-1}\right\}^2 + 4\left((\beta-1)D^2 + \widetilde{B}^*_{\bar{h}}\right) \tag{32}$$

Due to the linear dependence $\widetilde{B}^*$ on $\bar{h}$ in (31) the boundaries of the hyperbolicity domain $\bar{h}^{\pm} = \bar{h}^{\pm}(h_c)$ can be calculated from the quadratic equation $Disc^*(\bar{h},h_c) = 0$ relative to $\bar{h}$ in the explicit form:

$$\bar{h}^{\pm} = \frac{d_2 \pm q'\sqrt{d_3}}{d_1 \pm D'\sqrt{d_3}}$$

$$\text{where } \begin{cases} d_1 = \frac{1}{2}B^{\prime *}_{h_c} + (\beta-1)DD' \\ d_2 = (\beta-1)Dq' + \beta q D' - \frac{1}{2}\left(B_{h_c}(z^*,h_c) + B_{z^*}z^*_{h_c} - B^{\prime *}_{h_c}z^* - B^{\prime *}z^*_{h_c}\right) \\ d_3 = -\left((\beta-1)D^2 + B^{\prime *}\right) \end{cases} \tag{33}$$

VII. NUMERICAL RESULTS

For clarity, $h_c = 1$ is chosen as a reference solution. Fig.6 and Fig.7 represent the limits of the waves of maximum amplitude and the boundaries of hyperbolicity for some significant flow parameters. They show the influence of the fluid viscosity, and the bottom inclination respectively.

A. *Influence of the Fluid Viscosity:*

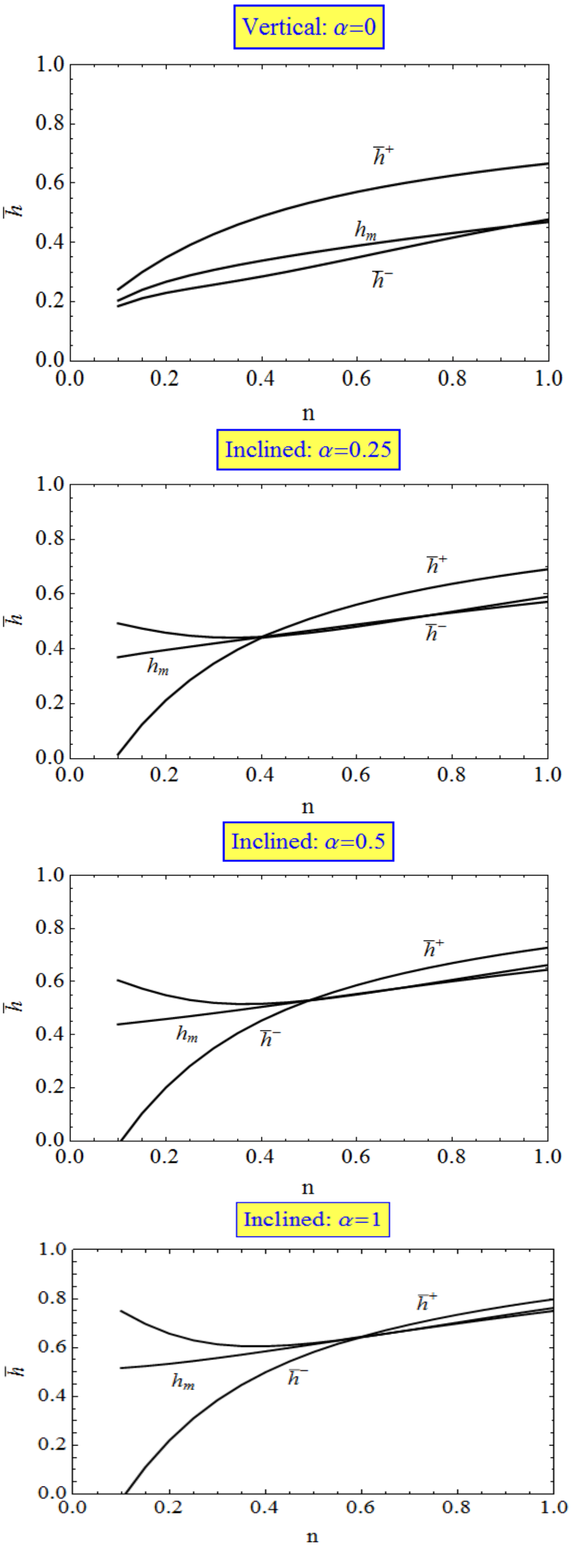
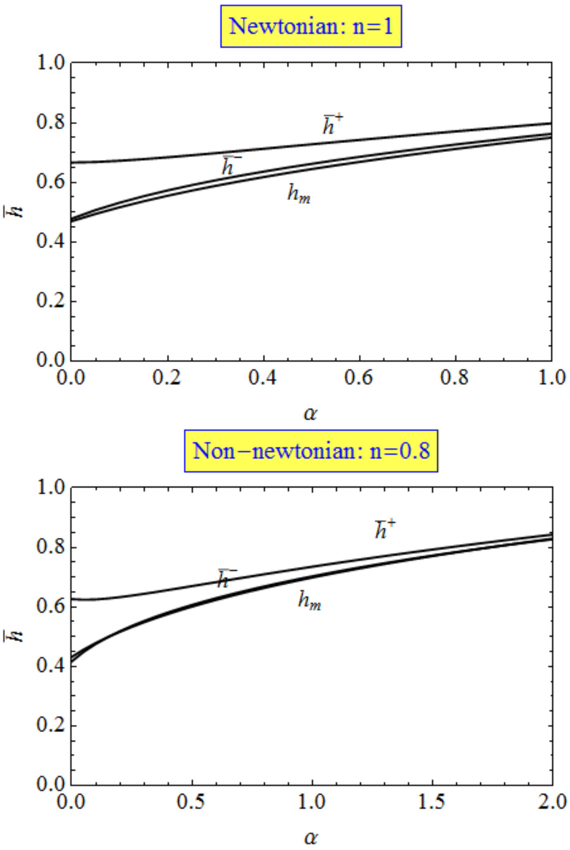


Fig.6 Limits of the waves of maximum amplitude h_m and boundaries of hyperbolicity $(\bar{h}^-, \bar{h}^+)$ as a function of n for $\alpha = 0,\ 0.25,\ 0.5,\ 1,$ and $h_c = 1$

From Fig.6, we note that, for a vertical flow, as the viscosity increases, the amplitude of admissible waves diminishes and the stability domain reduces until it disappears, that is to say, in the vertical case, the viscosity disfavors the production of RW; for an inclined flow, it is observed that as the viscosity increases up to a critical value, the stability domains diminish and eventually disappear, and after this critical value, the stability domains reappear, i.e., in the inclined case, the viscosity initially disfavors the production of RW until a critical value, after this critical value it favors the production of RW.

B. Influence of the Bottom Inclination:



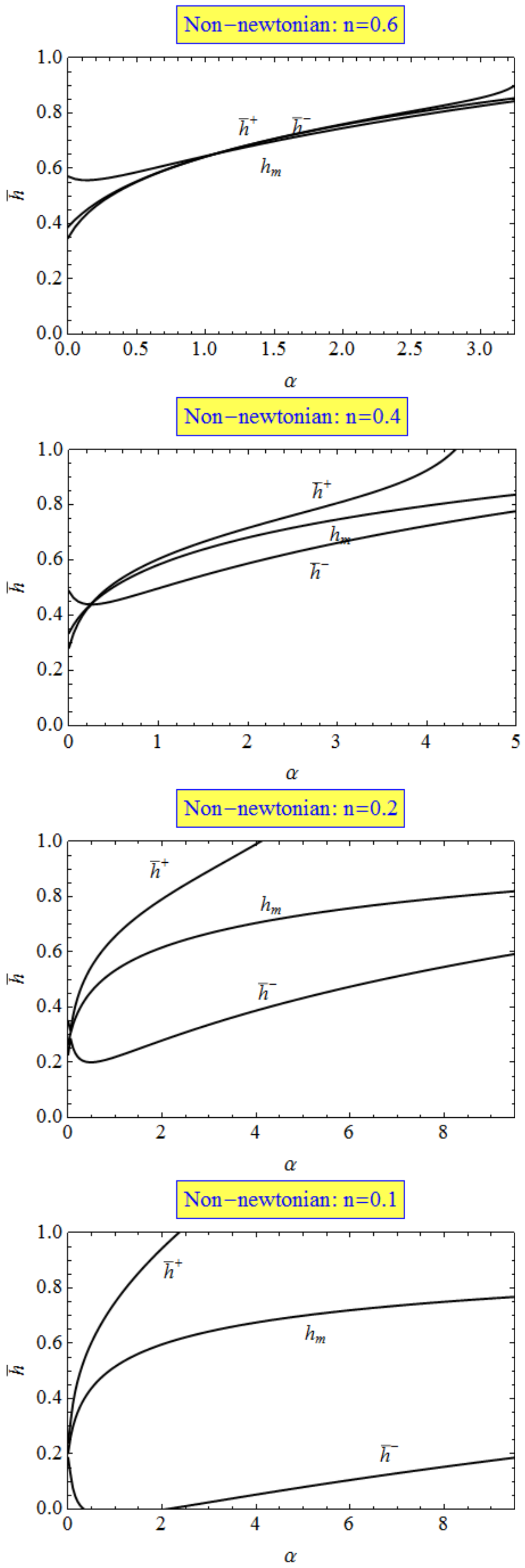
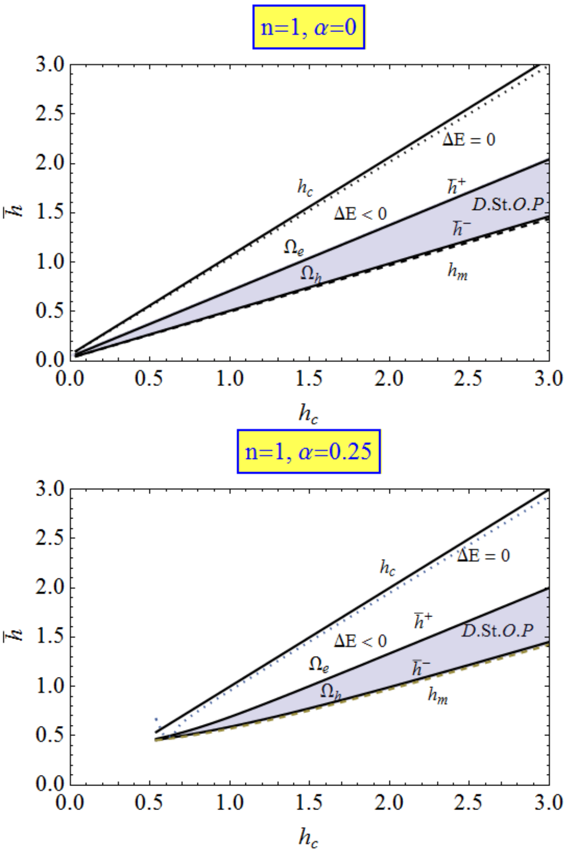


Fig.7 Limits of the waves of maximum amplitude h_m and boundaries of hyperbolicity $(\bar{h}^-, \bar{h}^+)$ as a function of α for $n = 1,\ 0.8,\ 0.6,\ 0.4,\ 0.2,\ 0.1$ and $h_c = 1$

From Fig.7, we can observe that, for a Newtonian fluid, as the bottom inclination decrease downwardly, the amplitude of admissible waves diminishes and the stability domain reduce until the disappearance, in other words, the bottom inclination disfavors the production of RW; for a slightly non-Newtonian fluid $n = 0.8$, it has the same tendency as the Newtonian fluid; for a highly non-Newtonian fluid $n = 0.6,\ 0.4,\ 0.2,\ 0.1$, it is found that as the bottom inclination decreases down to a critical value, the stability domains are reduced and disappear, after this critical value, the stability domains reappear and increase, i.e., the bottom inclination initially disfavor the production of RW until a critical value, and after this critical value, it favors the production of RW.

C. Nonlinear Stability Diagram

Let us vary h_c, the nonlinear stability diagrams corresponding to solutions (33) are represented in the $(h_c, \bar{h})$ – plane in Fig.8, where the curves $\bar{h} = \bar{h}^-$ and $\bar{h} = \bar{h}^+$ are the boundaries of stable hyperbolic domains Ω_h. In the elliptic domains Ω_e roll waves are unstable. The curve $\bar{h} = h_c$ corresponds to roll wave of infinitesimal amplitude, and the curve $\bar{h} = h_m$ corresponds to roll wave of admissible maximal amplitude. The curve $\Delta E = 0$ corresponds to the roll waves with zero dissipation across the shock [1]. Obviously the shock is accompanied by the loss of energy $\Delta E < 0$, the calculation of ΔE can be seen in [1].



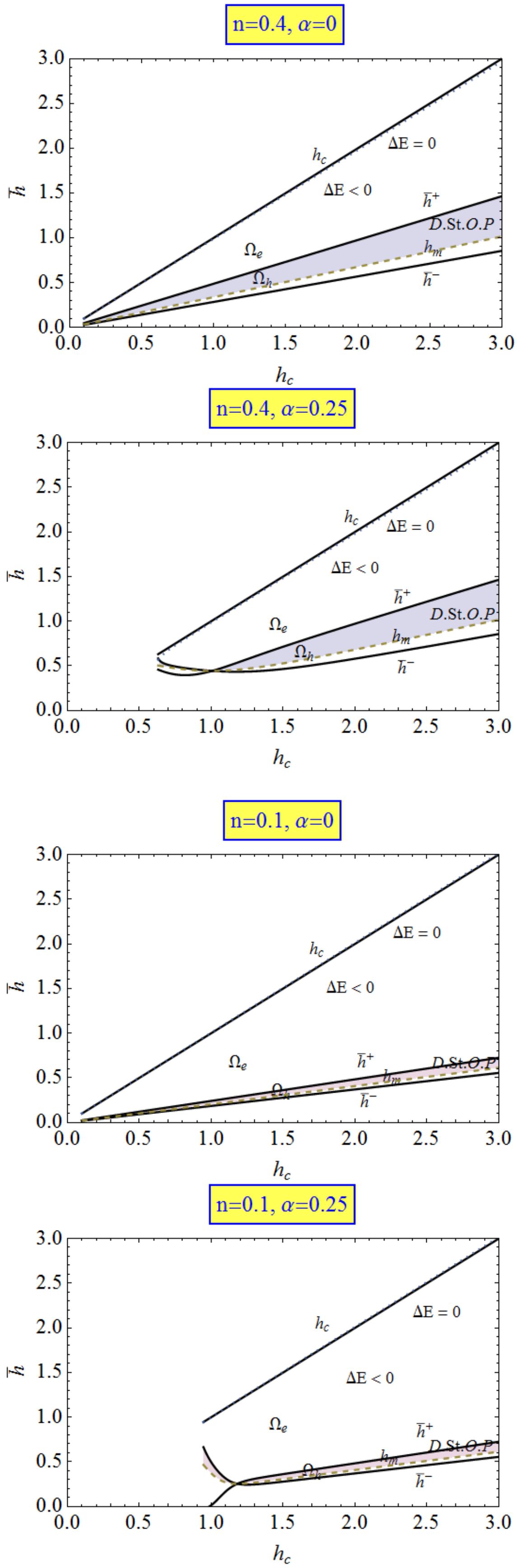


Fig.8 Nonlinear stability diagram of roll waves for different values of (n, α) -----: $\bar{h} = h_m$; ….: $\Delta E = 0$;
Gray region: Stable Periodic Wave Domain (D.St.O.P)

From Fig.8, we can see, the waves of moderate amplitude are stable, and the waves of small amplitude are unstable. The roll wave becomes stable only if they have the amplitude exceeding some critical value. The roll wave without dissipation across the shock is a wave of small amplitude. This type of wave is unstable. It makes sense that the waves of maximum amplitude are potentially damaging.

VIII. CONCLUSION

We have investigated the roll wave's generation on laminar flow of the thin layer down an inclined plane by nonlinear hyperbolic system [4], in which the rheological behavior is modeled by a power law. It has been shown that the roll wave's periodic solution can be described by two parameters analogously to roll waves in open channel flow. The linear stability criterion is unconditionally satisfied together with a roll wave required condition if the inequality (22) is verified, while dimensionless critical depth is less than 1. A stability criterion based on hyperbolicity of modulated equations has been presented. The asymptotic analysis has been performed and the stability criterion for roll waves of maximal amplitude as well as the approximate position of boundaries of the stability region has been derived. Numerical calculations have been performed for some significant flow parameters. They have revealed that for a critical value of α or n , the roll waves disappear, it can be very interesting in manufacturing industry.

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