

# An Algebraical Superposition Technic for Transformation From Z Domain to Time Domain

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**Abstract-** An algebraical superposition technic for transformation from z domain to time domain is presented. The establishing model process is: starting the inverse z transforms integral formula, and in its region of convergence based on the 'complex function integral' the inverse z transform integral is represented by  $2k-1$  term series. When the transform function on  $\pm k$  term series along integral circle are conjugated complex number distribution, the bidirectional series sum on  $k[-K, K]$  term series can be expressed by a monomial trigonometric function series sum on  $k[0, K]$ , in which the members are easy calculation and sum. In the paper the solution process and main points are presented. The application examples are shown, the results are supported to the algebraical superposition technic. The technic can be used to solve the problem which are difficult to be solved by presented method (such as Partial Fraction Exapration method, etc).

**Index Terms-** algebraical superposition technic, inverse z transform, conjugated complex, Partial Fraction Exapration method

## I. INTRODUCTION

In the fields of electronics, dynamics, controls and other fields of science and technology, the acquirement of system parameters is very important. Some system parameters vary with time, while some vary with frequency. The former is called time-domain parameter, and the latter is called frequency-domain parameter. We know one from another by the transformation between time domain and frequency domain. The signal variation with time often has two states: continuous-time variation and discrete-time variation, the later is often from sampling a continuous-time variation signal. With regard to time domain-frequency domain transform, Laplace transform is often used for continuous-time variation signal. In discrete-time signal system z transform is a strong tool to analysis linear time-invariant system. The z transform play an important role in solving difference equation, which is similar to that Laplace transform play an important role in solving differential equation [1][2][3].

For Laplace transform and z transform, to perform transform from time domain to frequency domain, some results can be obtained from transform-pair Table or from the numerical integral [1] [2]. For a inverse transform from frequency domain, z domain to time domain it is often necessary to use one kind of inverse transform formular, but its time domain solution may appear data including  $t < 0$  that is disagreement to "no meaning or no necessary for  $t < 0$ ". Maybe, it is a puzzle.

In the paper we reserch to use algebraical superposition technic to inverse z transform for extending the domain of inverse z transform

and finding easily out solution data only for  $t \geq 0$ . The mathematical mode for algebraical superposition technic and the main points for executive calculation will be shown in next two sections. Calculation examples will be exhibited to prove algebraical superposition technic in another section. It should point out that known method for inverse z transform are direct division, partial-fraction expansion, contour integration [1][2]. But those require the function formula to be rational function. The algebraical superposition technic in the paper is suit for both rational functions and irrational functions.

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## II. PRINCIPLE MODEL

We now enter into a principle to perform inverse z transform by algebraical superposition technic. Setting up process as follows:

For a causal signal to find out time domain sequences based on inverse z transform [3] is

$$x(n) = \frac{1}{j2\pi} \oint_c X(z) z^{n-1} dz, \quad n \geq 0 \quad (1)$$

Where  $n \geq 0$  is the symbol of causal signal,  $x(n) = 0$  for  $n < 0$ . The above integral path c is a closed circle with radius r on the convergence region of z transform. The region of convergence for transform in complex z-plane is located at the region where  $r > r_a$ ,  $r_a$  is convergence radius. Otherwise, the z transforms formula corresponding to equation (1) is in the following

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \quad (2)$$

To get the integral calculation for (1), "complex functions integral" [4] is used. For the integral calculation, the integral path c can be choosed to consist of  $2k$  small arc elements and corresponding equally divided points are  $z_0, z_1, z_2, \dots, z_{2k-1}$ . Suppose that  $f(z)$  is a simple value continue along the circumference c, when existing the following limit

$$\lim_{k \rightarrow \infty} \sum_{k=0}^{2K-1} f(z_{k+0.5})(z_{k+1} - z_k)$$

The integral can be expressed by the below sum

$$\oint_c f(z)dz = \lim_{K \rightarrow \infty} \sum_{k=0}^{2K-1} f(z_{k+0.5})(z_{k+1} - z_k) \quad (3)$$

Where the integral function and z point are all expressed in polar form:  $z_k = r_c \exp(j\phi_k)$ ,  $r_c$  is the radius of integral path circle ;  $\phi_k = k\pi / K$  is polar angle ;  $z_{k+0.5} = r_c \exp(j\phi_{k+0.5})$ , that is located in the center between  $z_k$  &  $z_{k+1}$ ,  $\phi_{k+0.5}$  is center polar angle . Substituting (3) into (1), we have

$$x(n) = \sum_{k=0}^{2K-1} I(k), \quad n \geq 0 \quad (4)$$

$$I(k) = \frac{1}{j2\pi} X(z_{k+0.5})z_{k+0.5}^{n-1}(z_{k+1} - z_k)$$

Uteriorly, observe above  $I(k)$  varies with  $k$ . We found that for many transform function  $X(z)$ , its two set of  $I(k)$  values on the integral path circle that are both along upper half-circle and along lower half-circle, are conjugated complex each other, *i.e.*, the following is hold

$$I(K+m) = I^*(K-1-m) \\ m = 0, 1, 2, \dots, K-1 \quad (4.1)$$

Where  $I^*$  is the conjugated complex of  $I$  When (4.1) hold, the whole integral values are double real part values, From (4) we obtain

$$x(n) = 2.0 \sum_{k=0}^{K-1} R_e[I(k)]. \quad n \geq 0$$

$$I(k) = \frac{1}{j2\pi} X(z_{k+0.5})z_{k+0.5}^{n-1}(z_{k+1} - z_k) \quad (5)$$

Another equivalent formula to (4) is

$$x(n) = \frac{1}{j2\pi} \sum_{k=0}^{2K-1} X(z_{k+0.5})z_{k+0.5}^{n-1}z_{k+1} \\ - \frac{1}{j2\pi} \sum_{k=0}^{2K-1} X(z_{k+0.5})z_{k+0.5}^{n-1}z_k, \quad n \geq 0 \quad (6)$$

Practical calculation show: the two sequence values respectively due to (4) and (6) is much closed. For example the real parts of two sets of data have agreement in 12 digits.

### III. EXECUTIVE MAIN POINTS

(1) The first step to find out the time domain sequence is to determine the region of convergence (ROC) of z transform in the complex z-plane. The ROC has often characters as follows:

- (a) The ROC is often a cirque form, where its center is the coordinate origin point.
- (b) No any polar point of  $X(z)$  is located inside the ROC.
- (c) In some tables of z transform, accompanying  $x(n)$ ,  $X(z)$  formula, the ROC are often listed.

- (d) If  $X(z)$  given is over presented materials, finding out the polar points by observing and testing, in the ROC any polar points should not be appeared.

(2) It is necessary to calculate and analyse  $I(k)$  variation with  $k$  by (4), for judging whether (4.1) hold or not. If (4.1) hold, the time domain sequence  $x(n)$  can be calculated by (5). If (4.1) was not, the time domain sequence  $x(n)$  can be calculated by (4) or (6).

(3) The main points to calculate the time domain sequence are: The calculations are occurred inside the ROC, *i.e.* the cirque is to meet  $r_c > |a|$ ,  $a$  is convergence radius. The integral path circle is often divided into  $2k$  small arc elements and corresponding center points are  $z_0, z_1, z_2, \dots, z_{2k-1}$ . Theoretically,  $k$  should approach to  $\infty$ , but in practical calculation a set of time domain sequence is first calculated or a given  $k_i$ , then a set of time domain sequence is calculated for  $k_{i+1}$ . Comparing the two sets of sequence for any time, if the two have same value in three digits or more, we can approximatively conclude that time domain sequence in  $k_{i+1}$  set of sequence is a expected one or called 'convergence value'.

(4) To reach the 'convergence value' of time domain sequence it is often needed to calculate and superpose using double precision calculation.

(5) It is necessary to validate the time domain solution sequence. Validation method is to transform the time sequence obtained into z domain sequence called 'calculation value'. At one time to calculate z domain sequence called 'theory value' from  $X(z)$  in z plane. Comparing the calculation value with the theory value, the conclusion will be reached.

### IV. APPLICATION EXAMPLES

Two examples will be exhibited. First one is to know its time domain solution, so it is easy using the time domain solution to validate the algorithm in the paper. Second one exhibits that the algorithm in the paper is also to fit for the transform of an irrational function in z region.

(4.1) Example 1

Z transform function  $X(z)$  formula is as follows

$$X(z) = \frac{z}{z - e^b},$$

$$\text{Convergence region: } |z| > |e^b| \quad (8)$$

In (8) b is a real constant. The time domain solution sequence is

$$x(n) = e^{bn} \quad (9)$$

So the theoretical value of example1 is from (9). It is first needed to determine b that is due to minimum radius of convergence region,  $|z| > |e^b| = r_{\min}$ . Here taking

$b=0.2$ ,  $e^b = 1.221403 = r_{\min}$ , therefor the integral path circular radius  $r_c > r_{\min} = 1.221403$ . The calculation show:  $r_c = 1.3 - 1.5$ , the output time domain solution sequence values have an agreement in 5, 6 digit.

In calculation (9), whole sequence number  $n_m = 20$  and  $r_c = 1.3$ ,  $K = 180$ ,  $b=0.2$ , the output time domain solution sequence is theoretical value. The comparison between the theoretical and calculative value is shown in Fig 1, the two are well agreement in 5 digits.

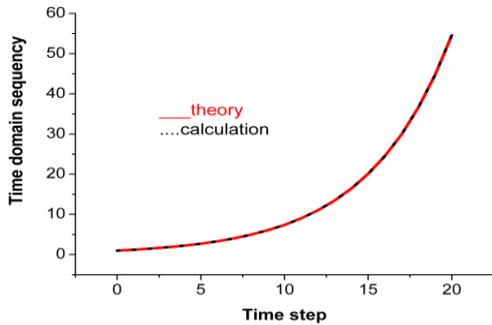


Fig.1 comparison between the theoretical and calculative value

(4.2) Example 2

Z transform function  $X(z)$  formula is as follows

$$X_s(z) = \sqrt{\frac{z}{z-2}} \frac{1}{z-1} \quad (10)$$

(10) contains an irrational fraction expression part that is difficult to solve by presented method. But its solving scheme can be found in the following (a).

(a) Z transform function  $X(z)$  formula analysis

Squared  $X_s(z)$  is a rational fraction expression  $X(z)$  as follows

$$\begin{aligned} X(z) &= X_s \times X_s = \sqrt{\frac{z}{z-2}} \frac{1}{z-1} \times \sqrt{\frac{z}{z-2}} \frac{1}{z-1} \\ (11) \quad &= \frac{z}{(z-2)(z-1)^2} \end{aligned} \quad (11)$$

Known  $X(z)$  having time domain solution

$$x(n) = 2^n - 1 - n \quad (12)$$

Next, we use symbol  $ZT$  to denote z transform, \* denote convolution, using convolution property of z transform we have

$$\begin{aligned} ZT[x(n)] &= ZT[x_s(n) * x_s(n)] = X_s \times X_s \\ &= \sqrt{\frac{z}{z-2}} \frac{1}{z-1} \times \sqrt{\frac{z}{z-2}} \frac{1}{z-1} \\ x(n) &= x_s(n) * x_s(n) \end{aligned} \quad (13)$$

It is necessary to point out  $x_s(n)$  corresponding to  $X_s(z)$  can be found by the algebraical superposition technic in the paper. So we can compare the two sequence respectively from (12) 、(13) to judge the algebraical superposition technic being good or not. The related calculation and comparison are in the following.

(b) Calculation and comparison

To comparison time domain solution sequence from (12) 、(13) may perform below steps. First, calculate  $I(k)$  distribution along integral circular path by (4) and find out the sequence by (5).

Second, work out convolution sequence  $x_s(n) * x_s(n)$  that is the calculation value. As the sequence values increase quickly with  $n$  to approach infinite, for comparison, only several points neighboring 0 were calculated..

We use calculative parameters  $r_c = 1.74$ ,  $k = 360$   $n = 1, 2, \dots, 8$  to find out sequence  $x_s(n)$ . The comparison for the calculation and theoretical sequence is shown in Fig.2. As time domain sequence value increase much fast with  $n$ , the maximum point number is only 8, and the calculative values are shown in their logarithm values. In practice at 2-8 point, the calculation values are : 0.26、1.26、4.15、10.4、24.5、55.2、122、276, the cooresponding theoretical values are: 0、1、4、11、26、57、120、247. It is obvious that the variation trend for theoretical and calculative values are similar, and two sets of numerical values are closed, but the approximate degree is less than example 1, as the convolution calculation is here entered.

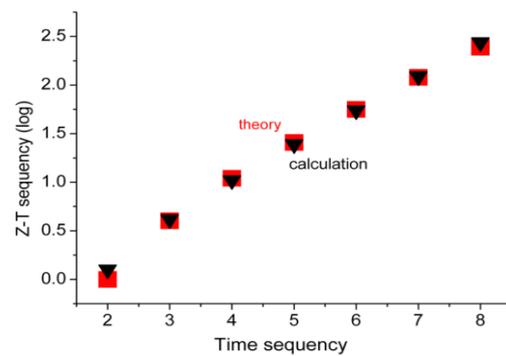


Fig2. The time domain solution sequence comparison between theory and calculation values for example 2.

V. SUMMARIES

(1) An algebraical superposition technic for trasformation from z domain to time domain is presented. The establishing model process is: astarting the inverse z transforms itergral formula, and in its region of convergence based on the 'complex function integral' the itergral is represented by 2k-1 term series. When the transform function on  $\pm k$  item series along integral circle are conjugated complex number distribution, the bidirectional series sum on  $k[-K, K]$  term series can be expressed by a monomial trigonomial function series sum on  $k[0, K]$ , that are easy calculation and sum. The application examples are supported to the algebraical superposition technic. The technic can be used to solve the problem which are difficult to be solved by presented method.

(2) The calculation also show : The algorithm in paper can be used for both rational partial fractions and irrational partial fractions in z plane.

(3) Beside the example calculation another calculations are also done for some z region function (such as  $e^z$ 、 $\ln(z)$ 、 $\sin(z)$ 、 $\cos(z)$ 、 $\sqrt{z}$ ) to test their  $I(k)$  variation with  $k$  by (4) on the integral path circle (upper half circle and lower half circle). The results show the time domain solution sequence corresponding to above z region function can be found by (5) i.e., the time domain solution sequence is  $t \geq 0$ .

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