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On the Role of Taylor Expansion with Exponential Nonlinearity in BVP

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***Abstract*— The finite difference method is employed to introduce a consistent nonlinear system of algebraic equations corresponding to the nonlinear boundary value problem (BVP). Taylor expansion is used as a linearization technique to introduce a linear algebraic system of equations approximating the nonlinear system. Solutions of the linearized system are taken as the initiation for the Newton's Raphson iteration when solving the nonlinear system. Application to Bratu's problem and similar problems with damping effects have illustrated the efficiency of the treatment. Two numerical examples with their graphical representation are given. The calculated results have illustrated the correctness of the treatment.**

***Index Terms*— Bratu's problem, BVP, Exponential nonlinearity, Finite difference method, Newton's Raphson.**

I. Introduction

This paper considers the use of the finite difference method for solving some types of nonlinear boundary value problems (exponential nonlinearity). It is well known that discretization of nonlinear boundary value problems produces nonlinear algebraic systems [1] – [8]. Solutions of nonlinear systems of algebraic equations is a problem in itself, [8]. There is no analytical technique which can be used to solve nonlinear algebraic systems and the numerical treatment is the suitable choice. Among the numerical techniques for solving nonlinear algebraic systems is the Newton's Raphson technique. The difficulty in using the Newton's method is the need to know good initiation in the sense of being very close to the exact solution (fixed point). The importance of boundary value problems appears from the huge list of publications and from the applications in which BVP appears. Also, BVP appears in the treatment of many problems in partial differential equations (eigen value problems when the separation of variables technique is used). The general functional formulation of boundary value problems can be written in the form, [2], [7]

$$F(t,u(t),u'(t),u''(t))=0\,,\quad a\le t\le b \qquad (1)$$

Subject to the boundary conditions of the form

$$\delta_1 u(a)+\delta_2 u'(a)+\delta_3 u(b)+\delta_4 u'(b)=r_1$$

$$\gamma_1 u(a)+\gamma_2 u'(a)+\gamma_3 u(b)+\gamma_4 u'(b)=r_2 \qquad (2)$$

where δ_i , γ_i and r_i are given constants

when the BVP is linear in the highest order derivative, it can be written in the form

$$-u''(t)+f(t,u(t),u'(t))=0 \qquad (3)$$

We restrict this work to the discretization of nonlinear BVP with exponent nonlinearity of the form

$$u''(t)+a(t)u'(t)+b(t)u(t)+\lambda e^{\mu u(t)}=f(t)\,,\quad 0\le t\le 1$$

(4)

Subject to Dirichlet boundary conditions of the form

$$u(0)=r_1,\quad u(1)=r_2 \qquad (5)$$

Such problems appear in physical, chemical and engineering application problems. Radiation, Bratu's problem are good examples for such classes of BVP, [1]. The Bratu's problem is used as a benchmark problem to test the accuracy of many numerical techniques.

An expansion of the exponential term in the form

$$e^{u(t)}=1+u(t)+\frac{1}{2!}u^2(t)+\frac{1}{3!}u^3(t)+\cdots \qquad (6)$$

Is used. Truncating this series up to the linear terms introduces a linearized problem corresponding to the original problem. Discretizing the linearized problem produces a linear system. The solution of the linearized system is taken as the initial step to solve the original nonlinear system. Two numerical examples are given to illustrate the treatment.

II. Material and Methods

The continuous domain $[0,1]$ is replaced by a grid as shown in figure (1)

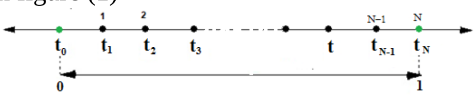

Figure 1 the grid imposed on the interval [0, 1]

The set of grid points denoted by

$$P_N=\{t_0,t_1,\cdots,t_N\}=\{t_i\}_{i=0}^{N}$$

$t_i=0+ih$, $i=0,1,\cdots,N$, with grid spacing $h=\frac{1-0}{N}$. It is natural to use the notation $u(t_i)=u_i$, the central difference approximation for the first order derivative

$$u'(t_i)=\frac{u_{i+1}-u_{i-1}}{2h}+O(h^2)\,, \qquad (7)$$

the central difference approximation for the second order derivative

$$u''(t_i)=\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+O(h^2)\,. \qquad (8)$$

It is generally accepted that every differential equation can be approximated by a corresponding finite difference scheme by replacing the derivative terms by their corresponding finite difference approximation at each grid point. Accordingly, equation (2) can be written in the discrete form

$$-\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+f(t_i,u_i,\frac{u_{i+1}-u_{i-1}}{2h})=0$$

$$i=1,\cdots,N-1 \qquad (9)$$

Accordingly, a system of algebraic equations is obtained the solution of the algebraic system gives approximation to the solution of the given boundary value problem. This system is

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linear when the function $f(t,u(t),u'(t))$ is linear otherwise it is nonlinear. Two practical problems with physical origins are considered.
The first is the Bratu type Boundary Value Problem.

One of the standard benchmark problems used to test the accuracy of many numerical methods is the Bratu's type equation. Bratu type equation in its simple form is

$$u''(t)-e^{-u(t)}=f(t)\,,\quad 0\le t\le 1$$
$$u(0)=r_1\,,\quad u(1)=r_2 \qquad (10)$$

Using the same grid described in figure (1) anyone can write the algebraic system in the form

$$-2u_1+u_2-h^2e^{-u_1}=h^2f(t_1)-r_1$$
$$u_{i+1}-2u_i+u_{i-1}-h^2e^{-u_i}=h^2f(t_i) \qquad (11)$$
$$i=2,3,\cdots,N-2$$
$$-2u_{N-1}+u_{N-2}-h^2e^{-u_{N-1}}=h^2f(t_{N-1})-r_2$$

This is a system of (N-1) nonlinear algebraic equations. The values required to initiate Newton's method can be calculated from the solution of linear system obtained from the truncated Taylor series for the exponential term, which has the form

$$-2u_1+u_2-h^2(1-u_1)=h^2f(t_1)-r_1$$
$$u_{i+1}-2u_i+u_{i-1}-h^2(1-u_i)=h^2f(t_i) \qquad (12)$$
$$i=2,3,\cdots,N-2$$
$$-2u_{N-1}+u_{N-2}-h^2(1-u_{N-1})=h^2f(t_{N-1})-r_2$$

This is a linear system and its solution is taken as the initiation for the solution of the nonlinear system and this can be seen from example (1)

The second case considers the existence of damping term. It is generally accepted that the first order derivative term has damping effects. So we consider a second order BVP with damping term in the form

$$u''(t)-u'(t)-e^{-u(t)}=f(t)\,,\quad 0\le t\le 1$$
$$u(0)=r_1\,,\quad u(1)=r_2 \qquad (13)$$

Using the same grid described in figure (1) the corresponding algebraic system is

$$-2u_1+(1-\frac{h}{2})u_2-h^2e^{-u_1}=h^2f(t_1)-(1+\frac{h}{2})r_1$$
$$(1+\frac{h}{2})u_{i-1}-2u_i+(1-\frac{h}{2})u_{i+1}-h^2e^{-u_i}=h^2f(t_i)$$
$$i=2,3,\cdots,N-2 \qquad (14)$$
$$(1+\frac{h}{2})u_{N-2}-2u_{N-1}-h^2e^{-u_{N-1}}=h^2f(t_{N-1})-(1-\frac{h}{2})r_2$$

This is a nonlinear algebraic system of (N-1) nonlinear algebraic equations. The values required to initiate Newton's method can be calculated from the solution of the linear system obtained from using Taylor series for the exponent term

$$-2u_1+(1-\frac{h}{2})u_2-h^2(1-u_1)=h^2f(t_1)-(1+\frac{h}{2})r_1$$
$$(1+\frac{h}{2})u_{i-1}-2u_i+(1-\frac{h}{2})u_{i+1}-h^2(1-u_i)=h^2f(t_i)$$
$$i=2,3,\cdots,N-2 \qquad (15)$$
$$(1+\frac{h}{2})u_{N-2}-2u_{N-1}-h^2(1-u_{N-1})=h^2f(t_{N-1})-(1-\frac{h}{2})r_2$$

This is a linear system and its solution is taken as the initiation for the solution of the nonlinear system (14) and this can be seen from example (2)

Theorem (1)

The finite difference representation of the BVP described in (11) or (14) is second order accurate.

Proof

The prove is straightforward by expanding the terms in standard Taylor series

Theorem (2)

The finite difference representation (11) is consistent the boundary value problem (10) and The finite difference representation (14) is consistent the boundary value problem (13) .

III. Numerical Examples

Two numerical examples are given to illustrate the theoretical behaviors described

Example 1

The nonlinear second order BVP

$$u''-e^{-u(t)}=2-e^{-t^2}$$
$$u(0)=0\,,\quad u(1)=1 \qquad (16)$$

Taking $h=0.1$ the nonlinear algebraic system can be written in the form

$$u_{i-1}-2u_i+u_{i+1}-h^2e^{-u_i}=h^2(2-e^{-t_i^2}),$$
$$i=1,\cdots,9 \qquad (17)$$
$$u_0=0, u_{10}=1$$

and this gives a non-linear system

$$-2u_1+u_2-0.01e^{-u_1}=0.0100995$$
$$u_1-2u_2+u_3-0.01e^{-u_2}=0.0103921$$
$$u_2-2u_3+u_4-0.01e^{-u_3}=0.0108607$$
$$\vdots \qquad (18)$$
$$-2u_9-0.01e^{-u_9}=-2.22757$$

The solution of the system (18) is shown in table (1) and figure (2).

Note that the coincidence of the approximate and the exact solutions due to the nature of the exact solution, $u(t)=t^2$

Table 1: The results of exact and approximation solutions

t_i	Exact	App
0.0	0	0
0.1	0.01	0.01
0.2	0.04	0.04
0.3	0.09	0.09
0.4	0.16	0.16
0.5	0.25	0.25
0.6	0.36	0.36
0.7	0.49	0.49
0.8	0.64	0.64
0.9	0.81	0.81
1.0	1	1

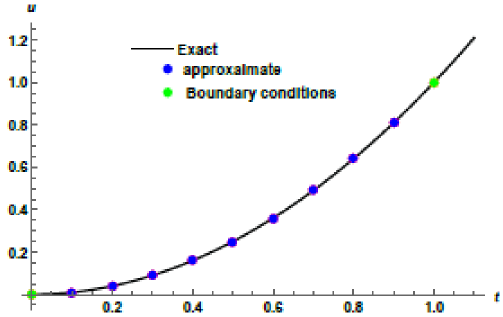

Fig 2: The behavior of the solution of nonlinear system when, h=0.1

Example 2

The nonlinear second order BVP with damping term

$$u''(t) - u'(t) - e^{-u} = 2 - 2t - e^{-t^2}$$
$$u(0) = 0, u(1) = 1 \qquad (19)$$

With the exact solution, $u(t) = t^2$

Taking $h = 0.125$ and using finite difference method, the system described in (14) becomes

$$\left(1+\frac{h}{2}\right)u_{i-1} - 2u_i + \left(1-\frac{h}{2}\right)u_{i+1} - h^2 e^{-u_i} = h^2\left(2 - 2t_i - e^{-t_i^2}\right),$$
$$u_0 = 0, u_8 = 1 \qquad (20)$$

and this gives a non-linear system

$$-2u_1 + 0.9375u_2 - 0.015625e^{-u_1} = 0.01196$$
$$1.0625u_1 - 2u_2 + 0.9375u_3 - 0.015625e^{-u_2} = 0.00875917$$
$$1.0625u_2 - 2u_3 + 0.9375u_4 - 0.015625e^{-u_3} = 0.00595601$$
(21)
$$1.0625u_6 - 2u_7 - 0.015625e^{-u_7} = -0.94086$$

The solution of the system (21) is shown in table (2) and figure (3).

Table 2: the results of exact and approximation solutions

t_i	Exact	App
0.0	0	0
0.1	0.015625	0.015625
0.2	0.0625	0.0625
0.3	0.140625	0.140625
0.4	0.25	0.25
0.5	0.390625	0.390625
0.6	0.5625	0.5625
0.7	0.765625	0.765625
0.8	1	1

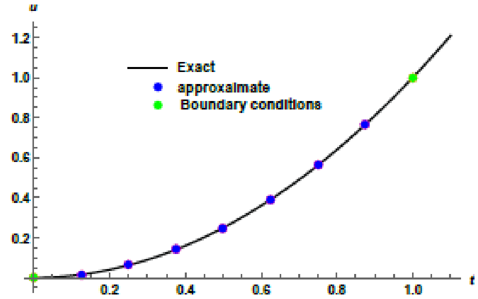

Fig 3: The behavior of the solution of nonlinear system when, h=0.125

IV. CONCLUSIONS

We considered nonlinear BVP with exponential nonlinearity with and without damping term. By using Taylor expansion with the exponential term and using the linear terms a linear BVP approximating the original nonlinear problem is obtained. The calculated results are in a good agreement with the exact solution due to the good initial values obtained from the solution of the linear problem. The technique can be used to other nonlinearities not only those of exponential forms and this will be our tasks in subsequent works.

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