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# A Non-Parametric Density Estimation Based Approach for Parameter Estimation of the Three-Parameter Burr XII Distribution

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***Abstract*— Burr XII distribution is widely used in different disciplines including engineering, business, besides others. A non-parametric density estimation based technique is applied to estimate the parameters of the three-parameter Burr XII distribution. A comparison between the proposed approach and the maximum likelihood estimators is performed. A Monte Carlo experiment of size 10000 is used to test the new estimator for different parameters of the true density. An improvement in the mean integrated square error as a measure of the closeness of the estimated density and the true density is noticed.**

***Index Terms*— Burr XII distribution, Non-parametric density, Kernel estimation, Cramer von Mises statistic.**

## I. Introduction

The Burr XII distribution has received a considerable amount of research since it was introduced in 1942 by I. W. Burr in Burr [2]. Applications of the distribution since it was introduced covered large extent of disciplines which include among other areas engineering, business, quality, mineralogy, as well as reliability and life testing. Researchers also showed interest in Burr distribution due to its relationships to other distributions which give the Burr distribution a flexibility to model many different applications. The first to introduce such relationships is Hatke [4] while Burr and Cislak [3] expanded the work of Hatke [4] to cover a wider spectrum of various distributions. In 1980 Tadikamalla [14] presented mathematical relations for Burr related distributions, where he showed that Lomax distribution is a special case of Burr XII distribution and the compound Weibull distribution is a generalization of the Burr distribution. In addition, he showed that Weibull, logistic, log logistic, normal, and lognormal distributions can be viewed as special cases of the Burr XII distribution through a proper choice of the parameters of the distribution. He also found different transformations that can be used to transform Burr XII to Burr III and Burr II to Burr XII (for details see Tadikamalla[14])). A number of authors have considered estimation of the parameters of the Burr XII distribution population. Zimmer et. al. [16] considered parameter estimation for the two parameter Burr XII distribution based on complete as well as censored samples with various sampling schemes. Watkins [15] used the maximum likelihood estimation (MLE) method to estimate the parameters of the three parameter Burr XII and compared the obtained model to a Weibull model for a given set odd at using a Discriminant statistic for the comparison.He concluded that Burr XII model is a better model compared to the Weibull model for the given data. In another study Alyousef [1] used the MLE technique to estimate the parameters of a two parameter Burr XII distribution from a doubly censored data. While Shao[11] generalized in a sense the approach introduced by Watkins [15] for both complete and censored samples. Jamjoom [7] estimated the parameters of the two parameter Burr XII distribution using the MLE and the least squares method. Olapade [9] suggested the six parameter Burr XII distribution through a generalization of the five parameter Burr XII distribution based on a transformation involving two independent random variables from exponential and gamma distributions. In our article the interest is to estimate the parameters of the three parameter Burr XII distribution based on small samples by minimizing a goodness of fit statistic. The approach decided on covering the varieties of shapes the distribution take depending on the values of the parameters considered. Section 2 discusses the solution of the ML equations. The method is based on numerically solving an equivalent nonlinear equation using an iterative scheme. The method is surveyed and the stopping rule is stated. In section 3 the application of a non-parametric density estimator to obtain estimates of the parameters of the three parameter Burr XII distribution is discussed. A Monte Carlo comparison of the maximum likelihood estimators and the minimum distance estimators is given. The integrated square error between the true density and the estimated true model for sample sizes of 5(5)20 for10000 Monte Carlo repetitions is used. A comparison is made between the estimators for various sample sizes (5(5)20) with different values of the three parameters.

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## II. MAXIMUM LIKELIHOOD ESTIMATORS FOR THE PARAMETERS OF THE 3 PARAMETER BURR XII DISTRIBUTION

The three parameter Burr XII distribution has a probability density function that takes the form:

$$f(x \mid \beta,k,\lambda)=\frac{\beta k}{\lambda}\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-(k+1)}\left(\frac{x}{\lambda}\right)^{\beta-1} \qquad ;x,\beta,k,\lambda>0$$

This form covers a variety of functional shapes depending on the values of the three parameters $\beta,k,\lambda$ respectively. A Burr XII distribution will be denoted by Burr($\beta,k,\lambda$), The following figure (Fig. 1) shows different Burr($\beta,k,\lambda$) shapes with various $\beta$ parameter values to cover the most variations in shape that Burr($\beta,k,\lambda$) takes depending on $\beta$.

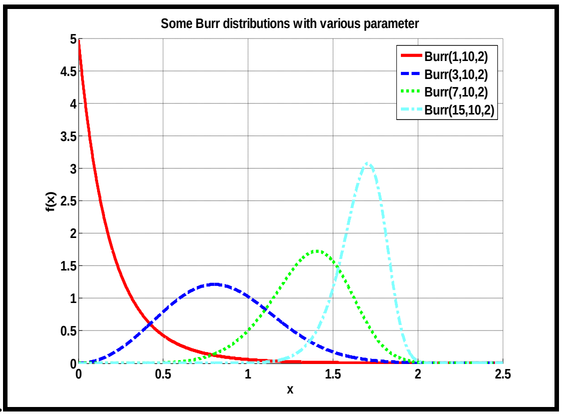


Fig. 1 The specrum of different shapes for three parameter Burr XII distribution Burr($\beta,k,\lambda$) for different $\beta$ values

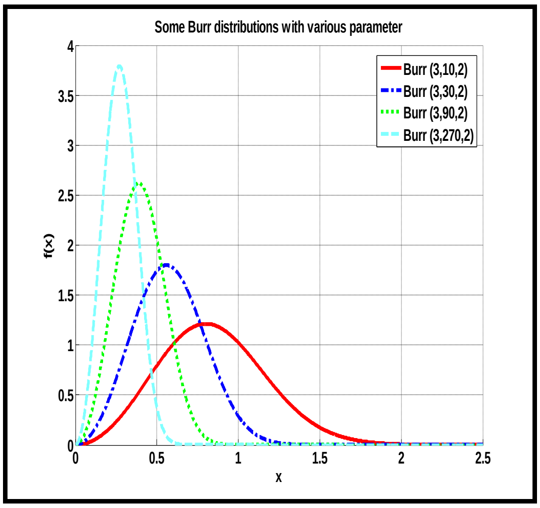


Fig. 2 The specrum of different shapes for three parameter Burr XII distribution Burr ($\beta,k,\lambda$) for different $k$ values.

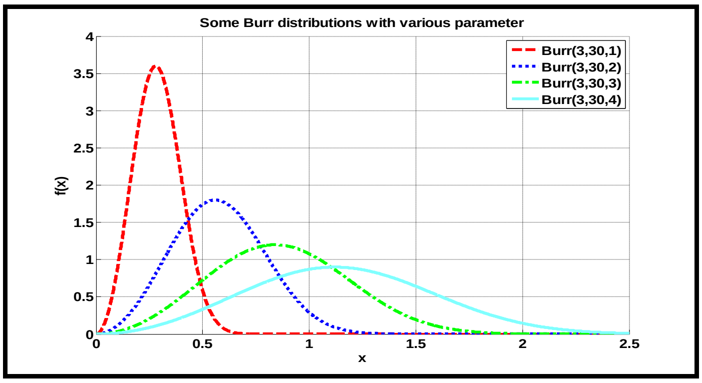


Fig. 3 The specrum of different shapes for three parameter Burr XII distribution Burr ($\beta,k,\lambda$) for different $\lambda$ values

The likelihood function of $f(x \mid \beta,k,\lambda)$ is given by:

$$L(\beta,k,\lambda \mid \underline{X})=\prod_{i=1}^{n}\frac{\beta k}{\lambda}\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{-(k+1)}\left(\frac{x_i}{\lambda}\right)^{\beta-1}$$

$$=\left(\frac{\beta k}{\lambda}\right)^{n}\left(\frac{\prod_{i=1}^{n}x_i}{\lambda}\right)^{\beta-1}\prod_{i=1}^{n}\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{-(k+1)}$$

The corresponding log likelihood function is:

$$L^{*}=\log L=n\left(\log\frac{\beta k}{\lambda}\right)+(\beta-1)\left(\sum_{i=1}^{n}\log(x_i)-\log\lambda\right)-(1+k)\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)$$

Hence, the maximum likelihood equations will be:

$$\frac{\partial L^{*}}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log(x_i)-\log\lambda-(1+k)\sum_{i=1}^{n}\left(\frac{1}{\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)}\right)\left(\left(\frac{x_i}{\lambda}\right)^{\beta}\right)\left(\log\left(\frac{x_i}{\lambda}\right)\right)=0$$

$$\frac{\partial L^{*}}{\partial k}=\frac{n}{k}-\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)=0$$

$$\frac{\partial L^{*}}{\partial\lambda}=-\frac{(n+\beta-1)}{\lambda}+(1+k)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\beta}\left(\frac{\beta}{\lambda}\right)\left(\frac{1}{\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)}\right)=0$$

The goal now is to solve the maximum likelihood equations simultaneously to find the estimates $\hat{\beta},\ \hat{k},\ and\ \hat{\lambda}$. Solving these equations is based on a modification of M.J.D. Powell's hybrid algorithm. This algorithm is a variation of Newton's method, which takes precautions to avoid large step sizes or increasing residuals. For further description, see More et al. [8].

Samples from the three parameter Burr XII distribution are generated using the inverse transformation technique for the distribution cumulative distribution function given by:

$$F(x)=1-\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-k} \quad ;x,\beta,k,\lambda>0$$

The subroutine RNSET from IMSL is used to initialize a random seed for use with a random number generator.

## III. MINIMUM DISTANCE ESTIMATION

Minimum distance estimation (MDE) is a method of Discriminant analysis based on a distance which can be used for continuous, discrete or mixed variables with known or unknown distributions. The method does not depend on one specific distance, so it is the investigator which has to decide the distance to be used according to the nature of the data. The method also makes use of the knowledge of prior probabilities and provides a numerical value of the confidence in the goodness of allocation of every data setunder study.

The method was first proposed by Wolfwitz in 1957. Parr and Schucany [10] demonstrated the robustness of MDE in predicting the location of the symmetric distributions. Hobbs, Moore, and James [5] used MDE to find the location of the gamma distribution. Similarly, Hobbs, Moore, and Miller [6] used MDE to estimate the location of the Weibull. Sultan [12] described a method for the calculation of the three-parameter Weibull distribution function from censored samples using MDE criteria.

In this section, it is required to estimate $\hat{\beta}, \hat{k}, and\ \hat{\lambda}$ for the 3-parameter Burr XII distribution such that a goodness of fit statistic is minimized. The Cramer von Mises statistics $W^2$ defined as:

$$W^2=n\int_{-\infty}^{\infty}\left[F_o(x)-\hat{F}(x)\right]^2 dF_o(x)$$

where $\hat{F}(x)$ is the empirical distribution function for the sample and $F_0(x)$ is a completely specified distribution function. The computational formula:

$$W^2=\sum_{i=1}^{n}\left[F_0\left(x_{(i)}\right)-\frac{i-0.5}{n}\right]^2+\frac{1}{12n}$$

is to be used. For this computational formula the non-parametric kernel estimator with a Gaussian kernel will be used to replace $\hat{F}(x)$ instead of the step function $\frac{i-0.5}{n}$. This kernel estimator will be:

$$\hat{F}(x)=\int_{-\infty}^{x}\frac{1}{nh}\sum_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-X_i}{h}\right)^2\right\}dx$$

$$=\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}h}\exp\left\{-\frac{1}{2}\left(\frac{x-X_i}{h}\right)^2\right\}dx$$

$$=\frac{1}{n}\sum_{i=1}^{n}\Phi\left(\frac{x-X_i}{h}\right)$$

where $\Phi(x)$ denotes the C.D.F for a standard normal random variable. Thus, the Cramer von Mises statistics $W^2$ will be:

$$W^2=\sum_{j=1}^{n}\left[F\left(X_{(j)}\right)-\frac{1}{n}\sum_{i=1}^{n}\Phi\left(\frac{X_{(j)}-X_{(i)}}{h}\right)\right]^2+\frac{1}{12n}$$

The optimal value of the window width h ( in the MISE sense) depends on the choice of the kernel K, the underlying unknown density f(x) and the sample size i.e.

$$h_{opt}=f_1(K).f_2\left(f(x)\right).f_3(n)$$

A reasonable approximation for this optimal value for basically a normal sample was suggested to be $h=kn^{-\frac{1}{5}}$ where k is a real constant. Although this approximation simplifies the optimal expression for the window width and works fine with the normal distribution it is not as good for other distributions. This leads to the idea of introducing the underlying density in another approximating expression for that h. The explicit expression for $h_{opt}$ is given as:

$$h_{opt}=m_2^{-2/5}\left\{\int K^2(t)dt\right\}^{1/5}\left\{\int f''^2(x)dx\right\}^{-1/5}n^{-1/5}$$

where:

$m_2$ denotes the kernel second moment which equals 1 in case of Gaussian kernel and $\int K^2(t)dt$ is simply equal to $\frac{1}{2\sqrt{\Pi}}$

Therefore

$$h_{opt}=\left(\frac{1}{2\sqrt{\Pi}}\right)^{\frac{1}{5}}\left\{\int f''^2(x)dx\right\}^{-\frac{1}{5}}n^{-\frac{1}{5}}$$

The development of the above formula for the Burr XII distribution is real lengthy and complicated in terms of finding the integral of the squared second derivative. An alternative for computing the window width which is more efficient computationally and gives a good improvement is to choose an empirical h which equals $sn^{-\frac{1}{5}}$ where s represents the sample standard deviation. This suggested h showed MISE which is close enough to the optimal theoretical and without a need to extensive computations.

The choice of the h parameter for the univariate case, can frequently be chosen visually in a satisfactory manner. The behavior of different distributions under a proposed choice of the h parameter had been studied in Moore and Sultan [13].The choice of h is data dependent which is a function of both the sample standard deviation and the sample size.

To evaluate the performance of the method a Monte Carlo experiment is designed. The inverse C.D.F technique (as mentioned in section 2.) is used for generating BURR XII random deviates of sample sizes 5(5)20 from ten different Burr($\beta, k, \lambda$) with various parameters to cover a wide span of shapes of the distribution. The data based choice of the smoothing parameter is calculated for each of 10000 different samples. The integrated square error ISE given as:

$$ISE = \int \left[ \hat{f}(x) - f(x) \right]^2 dx$$

is computed for each sample.

The estimation techniques used here include the MLE and the proposed method for the ten different set of chosen parameters. An estimate of the mean integrated square error MISE is obtained by averaging the ISE from the 10000 Monte Carlo repetitions for both estimation techniques. Likewise, an estimate of the standard deviation of MISE is computed (standard deviation is shown between brackets in italics in tables) and the results are given in the following tables (table 1. to table 10.)

Table 1: Results from Monte Carlo for BURR XII(3,30,2) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **0.544720** ***(0.466334)*** | **0.202548** ***(0.377298)*** |
| **10** | **0.351890** ***(0.338025)*** | **0.07531803** ***(0.126015)*** |
| **15** | **0.205944** ***(0.223296)*** | **0.04792036** ***(0.09288976)*** |
| **20** | **0.152186** ***(0.173282)*** | **0.03669148** ***(0.07058872)*** |

Table 2: Results from Monte Carlo for BURR XII(3,10,2) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **0.510601** ***(0.336927)*** | **0.136885** ***(0.248220)*** |
| **10** | **0.469155** ***(0.261774)*** | **0.07671752** ***(0.09680430)*** |
| **15** | **0.218857** ***(0.164992)*** | **0.04532990** ***(0.08676828)*** |
| **20** | **0.193021** ***(0.140677)*** | **0.03739739** ***(0.07081899)*** |

Table 3: Results from Monte Carlo for BURR XII(3,90,2) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **0.801687** ***(0.709705)*** | **0.306618** ***(0.565515)*** |
| **10** | **0.562406** ***(0.531487)*** | **0.115882** ***(0.187895)*** |
| **15** | **0.439533** ***(0.420888)*** | **0.08184177** ***(0.122786)*** |
| **20** | **0.298516** ***(0.325820)*** | **0.05752452** ***(0.101212)*** |

Table 4: Results from Monte Carlo for BURR XII(3,270,2) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **1.19108** ***(1.01558)*** | **0.459273** ***(0.853183)*** |
| **10** | **0.812075** ***(0.751275)*** | **0.185482** ***(0.318828)*** |
| **15** | **0.499699** ***(0.522184)*** | **0.125609** ***(0.254877)*** |
| **20** | **0.396965** ***(0.436069)*** | **0.09019919** ***(0.185015)*** |

Table5: Results from Monte Carlo for BURR XII(3,30,1) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **1.12011** ***(0.973649)*** | **0.431026** ***(0.799230)*** |
| **10** | **0.757022** ***(0.714205)*** | **0.167996** ***(0.292636)*** |
| **15** | **0.587841** ***(0.542831)*** | **0.120374** ***(0.187530)*** |
| **20** | **0.372518** ***(0.384577)*** | **0.09123140** ***(0.167794)*** |

Table 6: Results from Monte Carlo for BURR XII(3,30,3) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **0.504715** ***(0.327515)*** | **0.139340** ***(0.249265)*** |
| **10** | **0.321223** ***(0.217385)*** | **0.05724884** ***(0.104822)*** |
| **15** | **0.265678** ***(0.173472)*** | **0.04240599** ***(0.07426199)*** |
| **20** | **0.182958** ***(0.134752)*** | **0.03583250** ***(0.06901237)*** |

Table 7: Results from Monte Carlo for BURR XII(3,30,4) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **0.212859** ***(0.198331)*** | **0.09501764** ***(0.178328)*** |
| **10** | **0.122181** ***(0.126701)*** | **0.03938483** ***(0.05361764)*** |
| **15** | **0.07449433** ***(0.08363305)*** | **0.02663498** ***(0.03791310)*** |
| **20** | **0.05778327** ***(0.06674877)*** | **0.02192475** ***(0.02839188)*** |

Table 8: Results from Monte Carlo for BURR XII(1,10,2) for sample size5(5)20, # of Cases $\leq$ 1000

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **1.84998** ***(0.869931)*** | **1.811422** ***(3.01449)*** |
| **10** | **1.82750** ***(0.893112)*** | **1.40439** ***(3.08512)*** |
| **15** | **1.75817** ***(0.823043)*** | **1.06517** ***(2.56277)*** |
| **20** | **1.72744** ***(0.768595)*** | **0.796584** ***(2.32056)*** |

Table 9:Results from Monte Carlo for BURR XII(7,10,2) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **0.460673** ***(0.370288)*** | **0.171447** ***(0.416971)*** |
| **10** | **0.252189** ***(0.227956)*** | **0.05263787** ***(0.08210582)*** |
| **15** | **0.227696** ***(0.189396)*** | **0.03804009** ***(0.05411233)*** |
| **20** | **0.195485** ***(0.156004)*** | **0.03152648** ***(0.04107207)*** |

Table 10: Results from Monte Carlo for BURR XII(15,10,2) for sample size 5(5)20

| **Sample Size** | **MISE(MLE)** ***(STD.DEV)*** | **MISE(CVM)** ***(STD.DEV)*** |
|---|---|---|
| **5** | **1.16987** ***(0.757150)*** | **0.380510** ***(0.837787)*** |
| **10** | **0.644153** ***(0.502428)*** | **0.152516** ***(0.194873)*** |
| **15** | **0.517999** ***(0.397166)*** | **0.107408** ***(0.129575)*** |
| **20** | **0.370839** ***(0.302661)*** | **0.08224370** ***(0.103112)*** |

The Monte Carlo experiment shows a significant improvement of the method over the classical MLE method for the BURR XII distribution with the given ten different parameter sets covering the various distribution shapes.

The previous experiment could be considered as an investigation for the behavior of the proposed technique. The results showed an improvement in the MISE and hence it can be recommended to use the proposed technique for small sample sizes of 5(5)20 for which we limit our study. The procedure used can be described in the following three steps:

- Different samples from BURR XII distribution with a given set of parameters for different sample sizes were generated using the inverse transformation technique. The uniform random number was generated using the RNUN routine from the IMSL.
- The maximum likelihood estimators for the 3-parameters were computed using a quasi Newton method as discussed earlier.
- The CvM statistic with $\beta, k, \lambda$ as the decision vector and with the given constraints (parameter space) on the values of the parameters was minimized.
- The new parameter estimates were compared with those of maximum likelihood estimates using the ISE as a measure for the comparison.

## IV. AN EXAMPLE

The new technique shows a significant improvement over the MLE method. In this section an example will be given to show the performance of the method over some given original density parameters. The example will show the generated sample from the distribution with a given set of original parameters $\beta, k, \lambda$ together with MLE parameters and the proposed method parameters (shown as the kernel column in the table). The chosen window width h is given as well as the ISE from both methods of estimation. These values will be given in table 11, The corresponding graph of each density including the nonparametric density estimator will be given in figure Fig 4.

Table 11: Results for a sample of size 10 from BURR XII(3,10,2)

| The Sample | 1.04144 , 0.873057 , 1.30356 , 0.672713 1.27296 , 1.13557 , 0.793738 , 0.674266 0.846105 , 0.472608 | | | | |
|---|---|---|---|---|---|
| Parameter | *B* | *κ* | *λ* | h-value | ISE |
| Original | 3 | 10 | 2 | ——— | ——— |
| MLE | 3.629 | 11.630 | 2.897 | ——— | 0.558885 |
| Kernel | 2.934 | 17.680 | 2.469 | 0.193382 | 0.000183 |

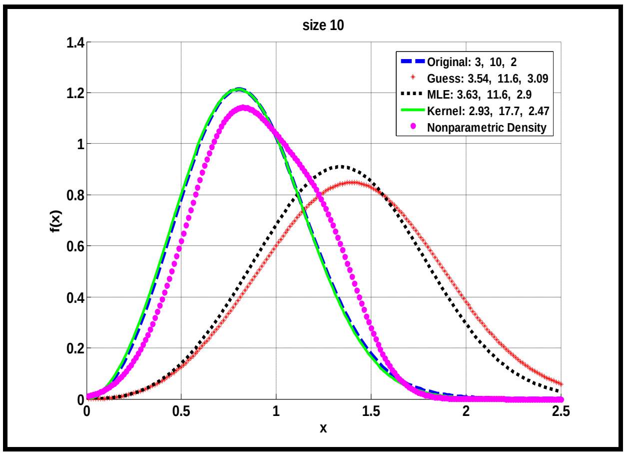


Fig. 4 Original Burr(3,10,2) with different estimated densities for sample size 10

The final conclusion is the minimum distance proposed estimation method using the CvMstatistic as a measure of the difference between a non-parametric estimator based on a suggested window width and a parametric density with unknown parameters gives in general a much smaller MISE value than the maximum likelihood method.

REFERENCES

[1] M. H. Alyousef, Estimation in doubly truncated Burr distribution, J. king Saud Univ., 14, 1-9, 2002
[2] I. W. Burr, Cumulative frequency functions, Ann. Math. Stat. 13, 215-232, 1942.
[3] I .W. Burr and P .J. Cislak, On a general system of distributions: I, Its curve-shaped characteristics; II, The sample median. Journal of the American Statistical Association, 63, 627-635, 1968.
[4] M .A Hatke, A certain cumulative probability function. Annals of Mathematical Statistics, 20, 461-463, 1949.
[5] J. R. Hobbs ; A. H. Moore and W. James, Minimum distance estimation of the three parameter gamma distribution. IEEE Transactions on Reliability 33 NO. 3: 237-240, 1984.
[6] J. R. Hobbs; A. H. Moore and R. M. Miller, Minimum distance estimation of the three parameter Weibull distribution. IEEE Transactions on Reliability 34 NO. 5: 495-496, 1985.
[7] A. A. Jamjoom, Some useful formulae for Monte Carlo simulations relating to Burr type II, XII distribution, Journal of applied sciences, 6, 553-558, 2006.
[8] More et. al., User guide for MINPACK-1, Argonne National Labs Report ANL-80-74, Argonne, Illinois, 1980.
[9] A.K. Olapade, On a six-parameter generalized Burr XII distribution, International Journal of Mathematics and Mathematical Sciences, 57, 3069-3074, 2004.
[10] W. C. Parr and W. R. Schucany, Minimum distance and robust estimation. JASA 75 NO 3 :616-624, 1980.
[11] Q. Shao, Notes on maximum likelihood estimation for the three-parameter Burr XII distribution. Comput. Statist. Data Anal. 45, 675–687, 2004.
[12] A. M. Sultan A data driven parameter estimation for the three parameter Weibull population from censored samples, Mathematical and Computational Applications, Vol. 13, No. 2, pp. 129-136, 2008
[13] A. M. Sultan and A. H. Moore, An application of non-parametric density estimation accepted publication in the 1990 joint national meeting, Anaheim, California
[14] P. R. Tadikamalla, A look at the Burr and related distributions. Int. Stat. Rev., 48, 337-344, 1980.
[15] A.J. Watkins, An algorithm for maximum likelihood estimation in the three parameter Burr XII distribution. Comput. Statist. Data Anal. 32,19–27, 1999.
[16] W.J. Zimmer; J.B Keats and F.K. Wang, The Burr XII distribution in reliability analysis. J. Qual. Tech. 30, 386–394, 1998.