Some Properties of Entire Functions Associated with L-entire Functions on C(I)

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Abstract—In this paper, let C(I) denote the Banach algebra of all continuous complex-valued functions defined on a close interval I in the set of real numbers, R. The functions having derivatives in the Lorch sense on the whole Banach algebra C(I) are considered and they are called L-entire functions [1, 3]. For each L-entire function on C(I), entire complex functions are associated and the relationship between their orders is studied. Even more, the possibility of locating the solutions of the equation F(f) = 0 from the location of zeros of the associated family of entire functions with F is analyzed too.

Index Terms—Banach algebras, locating zeros, order, L-entire functions, power series.

I. INTRODUCTION

Let I = [a, b] be a closed and bounded interval of R. Let C(I) denote the Banach algebra of continuous complex-valued functions defined on I, provided with the uniform convergence norm. The element 1_{C(I)} ∈ C(I) is called the unit element and it is the function satisfying 1_{C(I)}(t) = 1 for all t ∈ I.

A function F:C(I)→C(I) is said to have derivative in the Lorch sense, F'(f_0) at f_0, if for any > 0, a > 0 can be found such that for all h ∈ C(I) with ∥h∥ < δ,

\[ \|F(f_0 + h) - F(f_0) - hF'(f_0)\| \leq \|h\|\varepsilon. \]

If F has a derivative throughout a neighborhood of f_0, F is said to be a L- analytic function at f_0 and of course, if F is L-analytic in the whole C(I), it is said L-entire function on C(I), see [3].

If F is a L-entire function on C(I), by Theorem 26.4.1 of [3],

\[ F(f) = \sum_{n=0}^{\infty} g_n f^n, \quad f \in C(I), \]  

(1)

where g_n ∈ C(I) and \( \limsup_{n \to \infty} \|g_n\| = 0. \)

A L-entire function F on C(I) is associated with a family of entire complex functions, \( \{f_t\}_{t \in I} \) defined for each t ∈ I by

\[ f_t(z) = F(z1_{C(I)}(t)) = \sum_{n=0}^{\infty} g_n z^n, \quad z \in \mathbb{C}. \]

(2)

Also, it can be associated with the L-entire function F a function of complex variable, defined by

\[ g(z) = \int_{0}^{\infty} F(z1_{C(I)}(t))dt, \quad z \in \mathbb{C}. \]

(3)

By (1), for all \( z \in \mathbb{C}, \)

\[ g(z) = \sum_{n=0}^{\infty} \left( \int_{0}^{\infty} g_n(t)dt \right) z^n = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}, \]

(4)

and for all \( n \in \mathbb{N}, \)

\[ |a_n| = \left| \int_{0}^{\infty} g_n(t)dt \right| \leq (b - a) \|g_n\| \]

(5)

Inequality given in (5) implies that g is an entire function of complex variable.

Now, if F is a L-entire function on C(I), it is possible to find the relationship between the order of F and the orders of the entire functions \( f_t, t \in I \) and g, but all in all, there is not relationship between the orders of the entire functions \( f_t, t \in I \) and the order on the entire function g.

Furthermore, the possibility of locating the solutions of the equation \( F(f) = 0 \) from the location of the zeros of the equation \( f_t(z) = 0 \) will be analyzed.

II. ORDER OF A L-ENTIRE FUNCTION ON C(I)

The notion of order for an entire complex function has been extended without changes to entire functions defined from C, the complex number, onto a Banach space E, see [3]. This process can be done in the same way for a L-entire function on C(I), see [1].

Let F be a L-entire function on C(I). For each \( r > 0 \), it makes sense to define the quantity

\[ M(F,r) = \sup_{\|f\| = r} \|F(f)\|. \]

(5)
It is said that $F$ has finite order, if there are constants $\mu > 0$ and $\delta > 0$ such that

$$M(F, r) < e^{r^\mu}, \quad \text{if} \quad r > \delta$$

(6)

The lower bound of these $\mu$'s is called the order of $F$ and it will be denoted by $\rho(F)$.

In [1], it has shown that some relationships which are true for the order of an entire function of complex variable, are still maintained for the order of a L-entire function on $C(I)$, while others relationships are not longer fulfilled.

The next relationships is true and its proof can be found in [2].

$$\rho(F) = \limsup_{r \to \infty} \frac{\ln \ln M(F, r)}{\ln r}.$$  

(7)

Between the order of the L-entire function $F$ and the order of its associated entire function $f_r$ given in (2), there is the next relationship.

$$\rho(f_r) \leq \rho(F).$$  

(8)

for each $t \in I$. Indeed, as

$$|f_r(z)| \leq \|F(z1_{C(t)})\|$$

for all $z \in C$ and all $t \in I$. So,

$$M(f_r, r) \leq M(F, r).$$

and (8) follows from (7).

By the other hand, if $g$ is the entire function given in (4) associated with $F$, the inequality (5) gives

$$M(g, r) \leq M(F, r).$$

Thus, from (7)

$$\rho(g) \leq \rho(F).$$

(9)

Example 1. The inequality given in (9) can be strict, to see it, it is enough considered the L-entire function

$$F(f) = \sum g_n f^n.$$  

where $g_n(t) = \frac{t}{n!}$ for $t \in [-1,1]$. It is clear that $\rho(F) = 1$, while $\rho(g) = 0$.

Example 2. The inequalities given in (8) and (9) help to obtain information about the order of a L-entire function in cases where this quantity is impossible or difficult to calculate. For example, let $F$ be the L-entire function on $C([0,1])$.

$$F(f) = \sum g_n f^n.$$  

with

$$g_n(t) = \frac{n^3}{n^n} S_n(t), \quad t \in [0,1]$$

where $0 < \delta < 1$ and

$$S_n(t) = \left\{ \begin{array}{ll}
\frac{1}{n} - t & \text{if } t \in (0, \frac{1}{n}) \\
0 & \text{if } t \in \left[ \frac{1}{n}, 1 \right]
\end{array} \right.$$  

Then,

$$g(z) = \sum \frac{1}{n^n} z^n.$$  

and it is easy to see that $\rho(g) = \infty$. By (9), $\rho(F) = \infty$.

In general, the order of the entire function $f_r$, defined in (2), is not related to the order of the entire function $g$ defined in (3). For example, if $F$ is the L-entire function of the example 1, for all $t \in [-1,1]$, $\rho(f_r) = 1$ and $\rho(g) = 0$. So,

$$\rho(g) < \inf_{t \in [-1,1]} \rho(f_r).$$

By the other hand, if $F$ is the L-entire function of the example 2, for all $t \in [0,1]$, $f_r$ is a polynomial function with $\rho(f_r) = 0$ and $\rho(g) = \infty$. So,

$$\rho(g) > \sup_{t \in [0,1]} \rho(f_r).$$

III. LOCATION AND DISTRIBUTION OF THE ZEROS OF A L-ENTIRE FUNCTION ON $C(I)$

Let $D \subset C$ and $z \in C$. Let

$$\Omega^D = \{ h \in C(I) : h(I) \subset D \}$$

and

$$h_z(t) = z, \quad t \in I.$$  

If $z \in D$, then $h_z \in \Omega^D$.

The sets $\Omega^D$, have some properties whose proofs are obtained without difficulty from the functions $h_z$, with $z \in D$, such as those listed below.

1. $D$ is a convex set if and only if $\Omega^D$ is a convex set.
2. $D$ is a closed set if and only if $\Omega^D$ is a closed set.
3. $D$ is a bounded set if and only if $\Omega^D$ is a bounded set.
4. $D$ is an open set if and only if $\Omega^D$ is an open set.
5. $D$ is a compact set if and only if $\Omega^D$ is a compact set.
6. For $D_1 \subset \mathcal{C}$ and $D_2 \subset \mathcal{C}$, $\Omega^{D_1 \cap D_2} = \Omega^{D_1} \cap \Omega^{D_2}$.

In the following result, $f_t$ is the entire function of complex variable defined in (2) and $g$ is the entire function of complex variable defined in (3) and (4).

**Proposition 1.** Let $F$ be a L-entire function on $C(I)$ and $D \subset \mathcal{C}$. If $F(\Omega^D) \subset \Omega^D$, then $f_t(D) \subset D$ for all $t \in I$.

**Proof.** For $z \in D$, $h_z \in \Omega^D$ so $F(h_z) \in \Omega^D$. Now for all $t \in I$, $F(h_z(t)) \in D$, but
\[
F(h_z(t)) = \sum g_n(t)h_z(t)^n = \sum g_n(t)z^n = f_t(z).
\]
Since $z \in D$ is arbitrary, $f_t(D) \subset D$.

Generally it cannot enunciate a similar result for the entire function $g$ given in (4). However, under certain conditions over the set $D$, it is possible to enunciate some results involving $g$.

**Proposition 2.** Let $F$ be a L-entire function on $C(I)$ and let $D$ be a closed and convex subset of $\mathcal{C}$. If $F(\Omega^D) \subset \Omega^D$, then $g(D) \subset D$.

**Proof.** For $z \in D$, then $h_z \in \Omega^D$. If $t \in I$, $F(h_z(t)) \in D$. Taking $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ a partition of the interval $I = [a, b]$ by the convexity of $D$, for $s_i \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$,
\[
\sum (t_i - t_{i-1})F(h_z(s_i))
\]
is an element of $D$. Since
\[
g(z) = \int_0^1 F(z t_1 + t_0) dt = \int_0^1 F(h_z(t)) dt
\]
is the limit of sums of the type (10), together with the fact that $D$ is closed, it is concluded that $g(z) \in D$.

As a consequence of the Proposition 2 and the Schauder’s fixed-point theorem, see [5], the following corollary is obtained.

**Corollary 3.** Let $F$ be a L-entire function on $C(I)$ and let $D$ be a compact and convex subset of $\mathcal{C}$. If $F(\Omega^D) \subset \Omega^D$, then the entire function $g$ has a fixed point in $D$.

The following result provides information about the distribution and location of the zeros of a L-entire function on $C(I)$.

**Proposition 4.** Let $F$ be a L-entire function on $C(I)$ and let $D$ be a subset of $\mathcal{C}$. If the zeros of all entire functions $f_t, t \in I$, are in $D$, then the zeros of the L-entire function $F$ are in the set $\Omega^D$.

**Proof.** Taking $h \in C(I)$ and supposing $F(h) = 0$ but $h \not\in \Omega^D$, then exist $t_0 \in I$ such that $h(t_0) = z_0 \not\in D$. But
\[
f_t(z_0) = F(z_0 t_1 + t_0) = F(h(t_0)) = 0.
\]
Then $z_0 \in D$, which contradicts the assumption.

**Proposition 5.** Let $F$ be a L-entire function on $C(I)$ and $h \in C(I)$ a zero of $F$. Then $\dot{h}(t)$ is a zero of the entire function $f_t, t \in I$.

**Proof.** For fixed $t \in I$,
\[
f_t(h(t)) = F(h(t) t_1 + t_0) = F(h(t)) = 0,
\]
from here, the result is followed.

**Proposition 6.** Let $F$ be a L-entire function on $C(I)$ and $F(0) \neq 0$. Then $f_t(0) \neq 0$ for some $t \in I$.

**Proof.** Just look that
\[
f_t(0) = F(0 t_1 + t_0) = F(0(t))
\]

Using the Proposition 5 and 6, it is possible to prove, under certain conditions, that a L-entire function on $C(I)$ of finite order has a finite number of zeros in the closed ball with radius $r$ and center in the origin point.

Denote by $n(r)$ the number of zeros that a L-entire function $F$ has in the closed ball $\{h \in C(I) : |h| \leq r\}$. It is obvious that
\[
n(r) \geq \sup_{i \in I} n(r, f_t)
\]
where $n(r, f_t)$ is the number of zeros that the entire function $f_t$ has in the closed ball $\{z \in C : |z| \leq r\}$.

**Proposition 7.** Let $F$ be a L-entire function on $C(I)$ and let $\{h_k\}_{k \in \mathbb{N}}$ be the collection of zeros of $F$. Suppose $F(0) \neq 0$ and $h_k(t) \neq h_l(t)$ with $k \neq l$ and $t \in I$. Then $F$ cannot have infinitely many zeros in a ball of finite radius.

**Proof.** Since $F(0) \neq 0$, by Proposition 6, there is $t_0 \in I$ such that $f_{t_0}(0) \neq 0$. So $f_{t_0}$ is an entire function non-identically zero. By Proposition (5), $\{h_{t_0}(t)\}_{k \in \mathbb{N}}$ are the zeros of $f_{t_0}$ and since $h_k(t) \neq h_l(t)$ with $k \neq l$ then the zeros of $f_{t_0}$ are different.
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From here, $n(r, f_{h_k}) = n(r)$ and by Theorem 1.13.2 of [4], the conclusion is followed.

**Proposition 8.** Let $F$ be a L-entire function on $C(I)$ with $\rho(F) < \infty$. Let $\{h_k\}_{k \in \mathbb{N}}$ be the collection of zeros of $F$ where each one appears as many times as its multiplicity indicates. Suppose $F(0) \neq 0$ and $h_k(t) \neq h_l(t)$ with $k \neq l$ and $t \in I$. Then for each $r > 0$, the number $n(r) < \infty$.

**Proof.** Since $F(0) \neq 0$ by Proposition 6, there is $t_0 \in I$ such that $f_{h_0}(0) \neq 0$ and by Proposition (5), $\{h_k(t_0)\}_{k \in \mathbb{N}}$ are the zeros of $f_{h_k}$, and since $h_k(t_0) \neq h_l(t_0)$ with $k \neq l$, then the zeros of $f_{h_k}$ are different.

From here, $n(r, f_{h_k}) = n(r)$ and by Theorem 4.5.1 of [4], the conclusion is followed.

**REFERENCES**


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