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Some Properties of Entire Functions Associated with L-entire Functions on $C(I)$

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***Abstract*—In this paper, let $C(I)$ denote the Banach algebra of all continuous complex-valued functions defined on a close interval I in the set of real numbers, $\mathbb{R}$. The functions having derivatives in the Lorch sense on the whole Banach algebra $C(I)$ are considered and they are called L-*entire functions* [1, 3]. For each L-entire function on $C(I)$, entire complex functions are associated and the relationship between their orders is studied. Even more, the possibility of locating the solutions of the equation $F(f) = 0$ from the location of zeros of the associated family of entire functions with F is analyzed too.**

***Index Terms*—Banach algebras, locating zeros, order, L-entire functions, power series.**

I. Introduction

Let $I = [a, b]$ be a closed and bounded interval of $\mathbb{R}$. Let $C(I)$ denote the Banach algebra of continuous complex-valued functions defined on I, provided with the uniform convergence norm. The element $1_{C(I)} \in C(I)$ is called the unit element and it is the function satisfying $1_{C(I)}(t) = 1$ for all $t \in I$.

A function $F{:}C(I) \to C(I)$ is said to have derivative in the Lorch sense, $F'(f_0)$ at f_0, if for any $\varepsilon > 0$, a $\delta > 0$ can be found such that for all $h \in C(I)$ with $\|h\| < \delta$,

$$\|F(f_0 + h) - F(f_0) - hF'(f_0)\| \le \|h\|\varepsilon.$$

If F has a derivative throughout a neighborhood of f_0, F is said to be a L-analytic function at f_0 and of course, if F is L-analytic in the whole $C(I)$, it is said L-entire function on $C(I)$, see [3].

If F is a L-entire function on $C(I)$, by Theorem 26.4.1 of [3],

$$F(f) = \sum_{n=0}^{\infty} g_n f^n, \quad f \in C(I), \tag{1}$$

where $g_n \in C(I)$ and $\limsup_{n\to\infty} \|g_n\|^{\frac{1}{n}} = 0.$

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A L-entire function F on $C(I)$ is associated with a family of entire complex functions, $\{f_t\}_{t\in I}$ defined for each $t \in I$ by

$$f_t(z) = F(z1_{C(I)})(t) = \sum_{n=0}^{\infty} g_n z^n, \quad z \in \mathbb{C}. \tag{2}$$

Also, it can be associated with the L-entire function F a function of complex variable, defined by

$$g(z) = \int_a^b F(z1_{C(I)})(t)dt, \quad z \in \mathbb{C}. \tag{3}$$

By (1), for all $z \in \mathbb{C}$,

$$g(z) = \sum_{n=0}^{\infty}\left(\int_a^b g_n(t)dt\right)z^n = \sum_{n=0}^{\infty} a_n z^n \quad z \in \mathbb{C}, \tag{4}$$

and for all $n \in \mathbb{N}$

$$|a_n| = \left|\int_a^b g_n(t)dt\right| \le (b-a)\|g_n\|. \tag{5}$$

Inequality given in (5) implies that g is an entire function of complex variable.

Now, if F is a L-entire function on $C(I)$, it is possible to find the relationship between the order of F and the orders of the entire functions $f_t, t \in I$ and g, but all in all, there is not relationship between the orders of the entire functions $f_t, t \in I$ and the order on the entire function g.

Furthermore, the possibility of locating the solutions of the equation $F(f) = 0$ from the location of the zeros of the equation $f_t(z) = 0$ will be analyzed.

II. Order of a L-entire Function on $C(I)$

The notion of order for an entire complex function has been extended without changes to entire functions defined from C, the complex number, onto a Banach space E, see [3]. This process can be done in the same way for a L-entire function on $C(I)$, see [1].

Let F be a L-entire function on $C(I)$. For each $r > 0$, it makes sense to define the quantity

$$M(F,r) = \sup_{\|f\| \le r} \|F(f)\|.$$

It is said that *F* has *finite order*, if there are constants $\mu > 0$ and $\delta > 0$ such that

$$M(F,r) < e^{r^{\mu}}, \text{ if } r > \delta \quad (6)$$

The lower bound of these μ's is called the *order* of *F* and it will be denoted by $\rho(F)$.

In [1], it has shown that some relationships which are true for the order of an entire function of complex variable, are still maintained for the order of a L-entire function on *C*(*I*), while others relationships are not longer fulfilled.

The next relationships is true and its proof can be found in [2],

$$\rho(F) = \limsup_{r\to\infty} \frac{\ln\ln M(F,r)}{\ln r}. \quad (7)$$

Between the order of the L-entire function *F* and the order of its associated entire function f_t given in (2), there is the next relationship.

$$\rho(f_t) \le \rho(F), \quad (8)$$

for each $t \in I$. Indeed, as

$$|f_t(z)| \le \|F(z1_{C(I)})\|$$

for all $z \in \mathbb{C}$ and all $t \in I$. So,

$$M(f_t,r) \le M(F,r).$$

and (8) follows from (7).

By the other hand, if *g* is the entire function given in (4) associated with *F*, the inequality (5) gives

$$M(g,r) \le M(F,r).$$

Thus, from (7)

$$\rho(g) \le \rho(F). \quad (9)$$

Example 1. The inequality given in (9) can be strict, to see it, it is enough considered the L-entire function

$$F(f) = \sum g_n f^n,$$

where $g_n(t) = \frac{t}{n!}$ for $t \in [-1,1]$. It is clear that $\rho(F) = 1$, while $\rho(g) = 0$.

Example 2. The inequalities given in (8) and (9) help to obtain information about the order of a L-entire function in cases where this quantity is impossible or difficult to calculate. For example, let *F* be the L-entire function on *C*([0,1]),

$$F(f) = \sum g_n f^n,$$

with

$$g_n(t) = \frac{n^3}{n^{n^{\delta}}} S_n(t),\ t \in [0,1],$$

where $0 < \delta < 1$ and

$$S_n(t) = \begin{cases} 6t\left(\frac{1}{n} - t\right) & t \in \left[0, \frac{1}{n}\right], \\ 0 & t \in \left]\frac{1}{n}, 1\right] \end{cases}$$

Then,

$$g(z) = \sum \frac{1}{n^{n^{\delta}}} z^n,$$

and it is easy to see that $\rho(g) = \infty$. By (9), $\rho(F) = \infty$.

In general, the order of the entire function f_t defined in (2), is not related to the order of the entire function *g* defined in (3). For example, if *F* is the L-entire function of the example 1, for all $t \in [-1,1]$, $\rho(f_t) = 1$ and $\rho(g) = 0$. So,

$$\rho(g) < \inf_{t\in[-1,1]} \rho(f_t).$$

By the other hand, if *F* is the L-entire function of the example 2, for all $t \in [0,1]$, f_t is a polynomial function with $\rho(f_t) = 0$ and $\rho(g) = \infty$. So,

$$\rho(g) > \sup_{t\in[0,1]} \rho(f_t).$$

III. Location and Distribution of the Zeros of a L-entire Function on *C*(*I*)

Let $D \subset C$ and $z \in \mathbb{C}$. Let

$$\Omega^D = \{h \in C(I) : h(I) \subset D\}$$

and

$$h_z(t) = z,\ t \in I.$$

If $z \in D$, then $h_z \in \Omega^D$.

The sets Ω^D, have some properties whose proofs are obtained without difficulty from the functions h_z, with $z \in D$, such as those listed below.

1. *D* is a convex set if and only if Ω^D is a convex set.
2. *D* is a closed set if and only if Ω^D is a closed set.

3. D is a bounded set if and only if Ω^D is a bounded set.
4. D is an open set if and only if Ω^D is an open set.
5. D is a compact set if and only if Ω^D is a compact set.
6. For $D_1 \subset \mathbb{C}$ and $D_2 \subset \mathbb{C}$, $\Omega^{D_1} \cap \Omega^{D_2} = \Omega^{D_1 \cap D_2}$.

In the following result, f_t is the entire function of complex variable defined in (2) and g is the entire function of complex variable defined in (3) and (4).

Proposition 1. *Let F be a L-entire function on C(I) and* $D \subset \mathbb{C}$. *If* $F(\Omega^D) \subset \Omega^D$, *then* $f_t(D) \subset D$ *for all* $t \in I$.

Proof. For $z \in D$, $h_z \in \Omega^D$ so $F(h_z) \in \Omega^D$. Now for all $t \in I$, $F(h_z)(t) \in D$, but

$$F(h_z)(t) = \sum g_n(t)[h_z(t)]^n$$

$$= \sum g_n(t) z^n = f_t(z).$$

Since $z \in D$ is arbitrary, $f_t(D) \subset D$. □

Generally it cannot enunciate a similar result for the entire function g given in (4). However, under certain conditions over the set D, it is possible to enunciate some results involving g.

Proposition 2. *Let F be a L-entire function on C(I) and let D be a closed and convex subset of* $\mathbb{C}$. *If* $F(\Omega^D) \subset \Omega^D$, *then* $g(D) \subset D$.

Proof. For $z \in D$, then $h_z \in \Omega^D$. If $t \in I$, $F(h_z)(t) \in D$. Taking $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ a partition of the interval $I = [a,b]$, by the convexity of D, for $s_i \in [t_{i-1}, t_i]$, $i = 1,2,\ldots,n$,

$$\sum_{i=1}^{n} (t_i - t_{i-1}) F(h_z)(s_i) \tag{10}$$

is an element of D. Since

$$g(z) = \int_0^1 F(z 1_{C(I)})(t)dt = \int_0^1 F(h_z)(t)dt$$

is the limit of sums of the type (10), together with the fact that D is closed, it is concluded that $g(z) \in D$. □

As a consequence of the Proposition 2 and the Schauder's fixed-point theorem, see [5], the following collorary is obtained.

Collorary 3. *Let F be a L-entire function on C(I) and let D be a compact and convex subset of* $\mathbb{C}$. *If* $F(\Omega^D) \subset \Omega^D$, *then the entire function g has a fixed point in D.*

The following result provide information about the distribution and location of the zeros of a L-entire function on $C(I)$.

Proposition 4. *Let F be a L-entire function on C(I) and let D be a subset of* $\mathbb{C}$. *If the zeros of all entire functions* $f_t, t \in I$, *are in D, then the zeros of the L-entire function F are in the set* Ω^D.

Proof. Taking $h \in C(I)$ and supposing $F(h) = 0$ but $h \notin \Omega^D$, then exist $t_0 \in I$ such that $h(t_0) = z_0 \notin D$. But

$$f_{t_0}(z_0) = F(z_0 1_{C(I)})(t_0) = F(h)(t_0) = 0.$$

Then $z_0 \in D$, which contradicts the assumption. □

Proposition 5. *Let F be a L-entire function on C(I) and* $h \in C(I)$ *a zero of F. Then* $h(t)$ *is a zero of the entire function* $f_t, t \in I$.

Proof. For fixed $t \in I$,

$$f_t(h(t)) = F(h(t) 1_{C(I)}) = F(h)(t) = 0,$$

from here, the result is followed. □

Proposition 6. *Let F be a L-entire function on C(I) and* $F(0) \neq 0$. *Then* $f_t(0) \neq 0$ *for some* $t \in I$.

Proof. Just look that

$$f_t(0) = F(0 \cdot 1_{C(I)}) = F(0)(t).$$ □

Using the Proposition 5 and 6, it is possible to prove, under certain conditions, that a L-entire function on $C(I)$ of finite order has a finite number of zeros in the closed ball with radius r and center in the origin point.

Denote by *n(r)* the number of zeros that a L-entire function F has in the closed ball $\{h \in C(I) : \|h\| \leq r\}$. It is obvious that

$$n(r) \geq \sup_{t \in I} n(r, f_t),$$

where $n(r, f_t)$ is the number of zeros that the entire function f_t has the closed ball $\{z \in C : |z| \leq r\}$.

Proposition 7. *Let F be a L-entire function on C(I) and let* $\{h_k\}_{k \in \mathbb{N}}$ *be the collection of zeros of F. Suppose* $F(0) \neq 0$ *and* $h_k(t) \neq h_l(t)$ *with* $k \neq l$ *and* $t \in I$. *Then F cannot have infinitely many zeros in a ball of finite radius.*

Proof. Since $F(0) \neq 0$, by Proposition 6, there is $t_0 \in I$ such that $f_{t_0}(0) \neq 0$. So f_{t_0} is an entire function non-identically zero. By Proposition (5), $\{h_k(t_0)\}_{k \in \mathbb{N}}$ are the zeros of f_{t_0} and since $h_k(t) \neq h_l(t)$ with $k \neq l$ then the zeros of f_{t_0} are different.

From here, $n(r,f_{t_0}) = n(r)$ and by Theorem 1.13.2 of [4], the conclusion is followed. $\square$

Proposition 8. *Let F be a L-entire function on C(I) with* $\rho(F) < \infty$. *Let* $\{h_k\}_{k\in\mathbb{N}}$ *be the collection of zeros of F where each one appears as many times as its multiplicity indicates. Suppose* $F(0) \neq 0$ *and* $h_k(t) \neq h_l(t)$ *with* $k \neq l$ *and* $t \in I$. *Then for each* $r > 0$, *the number* $n(r) < \infty$.

Proof. Since $F(0) \neq 0$ by Proposition 6, there is $t_0 \in I$ such that $f_{t_0}(0) \neq 0$ and by Proposition (5), $\{h_k(t_0)\}_{k\in\mathbb{N}}$ are the zeros of f_{t_0}, and since $h_k(t_0) \neq h_l(t_0)$ with $k \neq l$, then the zeros of f_{t_0}, are different.

From here, $n(r,f_{t_0}) = n(r)$ and by Theorem 4.5.1 of [4], the conclusion is followed. $\square$

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