An elementary Proof of Hirschhorn's Conjecture

Jing Jin

***Abstract*—In this note, we give an elementary proof of Hirschhorn's formulas on the 4-dissections of Ramanujan's continued fraction R(q) and R^{-1}(q) which were conjectured by Hirschhorn and confirmed by Lewis and Liu.**

***Index Terms*—Ramanujan's continued fraction, Jacobi triple product identity.**

I. INTRODUCTION

Throughout this paper, we let $|q|<1$. We use the standard notation

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$$

and often write

$$(a_1,a_2,\cdots,a_n;q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty .$$

Recall that Ramanujan's continued fraction is defined by

$$R(q) := 1+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\cdots}.$$

Rogers [6] has proved that

$$R(q)=\frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}.$$

The 2-dissections and 5-dissections of R(q) and R^{-1}(q) were established by Andrews [1] and Hirschhorn [3]. Hirschhorn [3] also conjectured the 4-dissections of R(q) and R^{-1}(q), which can be stated as follows.

Conjecture 1.1 We have

$$R(q)=\frac{(q^8,q^{12};q^{20})_\infty}{(q^{20};q^{40})_\infty^4}\left(\frac{(q^8,q^{32};q^{40})_\infty (q^{32},q^{48};q^{80})_\infty^2}{(q^{16},q^{24};q^{40})_\infty^2}\right.$$
$$+q^6\frac{(q^8,q^{32};q^{40})_\infty (q^8,q^{72};q^{80})_\infty}{(q^4,q^{36};q^{40})_\infty^2}+q\frac{(q^{12},q^{28};q^{40})_\infty^2}{(q^4,q^{16};q^{20})_\infty}$$
$$\left.-q^3\frac{(q^8,q^{32};q^{40})_\infty^2}{(q^4,q^{16};q^{20})_\infty}\right) \qquad (1.1)$$

and

$$R^{-1}(q)=\frac{(q^{16},q^{24};q^{40})_\infty^2 (q^4,q^{36};q^{40})_\infty}{(q^{20};q^{40})_\infty^4}\left(\frac{(q^{24},q^{56};q^{80})_\infty^2}{(q^{12},q^{28};q^{40})_\infty}\right.$$
$$\left.+q^2\frac{(q^{16},q^{64};q^{80})_\infty^2}{(q^8,q^{32};q^{40})_\infty^2}\right)-q\frac{(q^4,q^{36};q^{40})_\infty (q^{16},q^{24};q^{40})_\infty^3}{(q^{20};q^{40})_\infty^4 (q^8,q^{12};q^{20})_\infty}$$
$$+q^7\frac{(q^4,q^{36};q^{40})_\infty^3 (q^{16},q^{24};q^{40})_\infty}{(q^{20};q^{40})_\infty^4 (q^8,q^{12};q^{20})_\infty}. \qquad (1.2)$$

(In fact, Hirschhorn's original formula on R^{-1}(q) has a typo, here we correct it.) Lewisand Liu [5] proved this conjecture by using an identity appeared in [4].

In this note, we give an elementary proof of Conjecture 1.1. Our method is different from Lewis and Liu's method, we only use the well-known Jacobi triple product identity [2]

$$\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\binom{n}{2}}=(z,q/z,q;q)_\infty . \qquad (1.3)$$

II. ELEMENTARY PROOF OF CONJECTURE 1.1

In this section, we give an elementary proof of Conjecture 1.1. Since the proofs of (1.1)
and (1.2) are similar, we only give the proof of (1.1).

Proof. We first derive the 2-dissection of R(q). By (1.3), we have

$$R(q)=\frac{(q^8,q^{12};q^{20})_\infty (-q,q^3,q^7,-q^9,q^{10},q^{10};q^{10})_\infty}{(q^4,q^6,q^{10},q^{10};q^{10})_\infty}$$
$$=\frac{(q^8,q^{12};q^{20})_\infty}{(q^4,q^6,q^{10},q^{10};q^{10})_\infty}\sum_{m,n=-\infty}^{\infty}(-1)^n q^{5m^2-4m+5n^2-2n}. \qquad (2.1)$$

Also, by (1.3), we see that

$$\sum_{m,n=-\infty}^{\infty}(-1)^n q^{5m^2-4m+5n^2-2n}$$
$$=\sum_{m\equiv n \pmod 2}(-1)^n q^{5m^2-4m+5n^2-2n}+\sum_{m\not\equiv n \pmod 2}(-1)^n q^{5m^2-4m+5n^2-2n}$$
$$=\sum_{k,j=-\infty}^{\infty}(-1)^{k-j} q^{10k^2+10j^2-6k-2j}+\sum_{k,j=-\infty}^{\infty}(-1)^{k-j} q^{10k^2+4k+10j^2+8j+1}$$
$$=\sum_{k=-\infty}^{\infty}(-1)^k q^{10k^2-6k}\sum_{j=-\infty}^{\infty}(-1)^j q^{10j^2-2j}$$
$$+q\sum_{k=-\infty}^{\infty}(-1)^k q^{10k^2-4k}\sum_{j=-\infty}^{\infty}(-1)^j q^{10j^2-8j}$$
$$=(q^4,q^{16},q^{20};q^{20})_\infty (q^8,q^{12},q^{20};q^{20})_\infty$$
$$+q(q^6,q^{14},q^{20};q^{20})_\infty (q^2,q^{18},q^{20};q^{20})_\infty . \qquad (2.2)$$

It follows from (2.1) and (2.2) that

Jing Jin, School of Mathematics, Nanjing Normal University Taizhou College, Taizhou, Jiangsu, P.R. China,.

$$R(q)=\frac{\left(q^8,q^8,q^{12},q^{12};q^{20}\right)_\infty}{\left(q^6,q^{10},q^{10},q^{14};q^{20}\right)_\infty}+q\frac{\left(q^2,q^8,q^{12},q^{18};q^{20}\right)_\infty}{\left(q^4,q^{10},q^{10},q^{16};q^{20}\right)_\infty}. \quad (2.3)$$

Now we turn to compute the 4-dissections of R(q). By (1.3), it is easy to verify that

$$\frac{1}{\left(q^3,q^5,q^5,q^7;q^{10}\right)_\infty}=\frac{\left(-q^3,-q^5,-q^5,-q^7,q^{10},q^{10};q^{10}\right)_\infty}{\left(q^6,q^{10},q^{10},q^{14};q^{20}\right)_\infty\left(q^{10};q^{10}\right)_\infty^2}$$

$$=\frac{1}{\left(q^6,q^{10},q^{10},q^{14};q^{20}\right)_\infty\left(q^{10};q^{10}\right)_\infty^2}\sum_{m,n=-\infty}^{\infty}q^{5m^2-2m+5n^2}. \quad (2.4)$$

Similarly, it follows from (1.3) that

$$\sum_{m,n=-\infty}^{\infty}q^{5m^2-2m+5n^2}=\sum_{m\equiv n \pmod 2}q^{5m^2-2m+5n^2}+\sum_{m\not\equiv n \pmod 2}q^{5m^2-2m+5n^2}$$

$$=\sum_{k,j=-\infty}^{\infty}q^{10k^2+10j^2-2k-2j}+q^3\sum_{k,j=-\infty}^{\infty}q^{10k^2+10j^2+8k+8j}$$

$$=\left(-q^8,-q^{12},q^{20};q^{20}\right)_\infty^2+q^3\left(-q^2,-q^{18},q^{20};q^{20}\right)_\infty^2. \quad (2.5)$$

In view of (2.4) and (2.5), we find that

$$\frac{1}{\left(q^6,q^{10},q^{10},q^{14};q^{20}\right)_\infty}$$

$$=\frac{\left(-q^{16},-q^{24},q^{40};q^{40}\right)_\infty^2+q^6\left(-q^4,-q^{36},q^{40};q^{40}\right)_\infty^2}{\left(q^{12},q^{20},q^{20},q^{28};q^{40}\right)_\infty\left(q^{20};q^{20}\right)_\infty^2}. \quad (2.6)$$

On the other hand, it follows from (1.3) that

$$\frac{\left(q,q^9;q^{10}\right)_\infty}{\left(q^5,q^5;q^{10}\right)_\infty}=\frac{\left(q,q^9,q^{10};q^{10}\right)_\infty\left(-q^5,-q^5,q^{10}\right)_\infty}{\left(q^{10};q^{20}\right)_\infty^2\left(q^{10};q^{10}\right)_\infty^2}$$

$$=\frac{1}{\left(q^{10};q^{20}\right)_\infty^2\left(q^{10};q^{10}\right)_\infty^2}\sum_{m,n=-\infty}^{\infty}(-1)^m q^{5m^2-4m+5n^2}. \quad (2.7)$$

Also, by (1.3), we see that

$$\sum_{m,n=-\infty}^{\infty}(-1)^m q^{5m^2-4m+5n^2}$$

$$=\sum_{m\equiv n \pmod 2}(-1)^m q^{5m^2-4m+5n^2}+\sum_{m\not\equiv n \pmod 2}(-1)^m q^{5m^2-4m+5n^2}$$

$$=\sum_{k,j=-\infty}^{\infty}(-1)^{k+j}q^{10k^2+10j^2-4k-4j}-q\sum_{k,j=-\infty}^{\infty}(-1)^{k+j}q^{10k^2+10j^2+6k+6j}$$

$$=\left(q^6,q^{14},q^{20};q^{20}\right)_\infty^2-q\left(q^4,q^{16},q^{20};q^{20}\right)_\infty^2. \quad (2.8)$$

Thus,

$$\frac{\left(q^2,q^{18};q^{20}\right)_\infty}{\left(q^{10},q^{10};q^{20}\right)_\infty}$$

$$=\frac{\left(q^{12},q^{28},q^{40};q^{40}\right)_\infty^2-q^2\left(q^8,q^{32},q^{40};q^{40}\right)_\infty^2}{\left(q^{20};q^{40}\right)_\infty^2\left(q^{20};q^{20}\right)_\infty^2}. \quad (2.9)$$

Combining (2.3), (2.6) and (2.9), we obtain (1.1). This completes the proof.

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